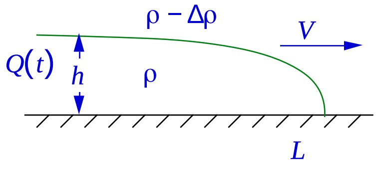
6. GRAVITY CURRENTS AND LAVA DOMES

1. High Reynolds number gravity currents have already been partially discussed.



$Fr = V/(g'h)^{1/2}$

$= \text{constant}$

generally 1 equation in 2 unknowns

How does the current spread with time? Can be partially determined by force balance (H², 1982)

two dimensional | axisymmetric

volume conservation

order of magnitude (two dimensional): $hL \sim \int_0^t Q(t')\,dt' \sim qt^{\alpha}$

constant (axisymmetric): $hR^2 \sim Qt^{\alpha}$

$\alpha = 0$ constant volume and $\alpha = 1$ constant flux

total inertial force

$F_i = \rho \int u u_x \,dx\,dy\,dz \sim \rho U^2 hw$ ← width

$\sim \rho q L w t^{\alpha-2}$

if $U \sim Lt^{-1}$

$F_i \sim \rho U^2 hR$

$\sim \rho Q R t^{\alpha-2}$

$U \sim Rt^{-1}$

buoyancy force

$F_g = \int \frac{\partial p}{\partial x} dx dy dz$

$\sim -\rho g' \int \frac{\partial h}{\partial x}\, dx\, dy\, dz$

$\sim \rho g' h^2 w \quad \sim \left(\rho g' q^2 w / L^2\right) t^{2\alpha}$

$F_g \sim \rho g' h^2 R$

$\sim \left(\rho g' Q^2 / R^3\right) t^{2\alpha}$

high Reynolds number, buoyancy-inertia balance $F_i \sim F_g$

$L \sim (g'q)^{1/3}\, t^{(\alpha+2)/3}$

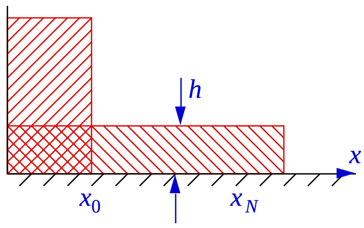
$R \sim (g'Q)^{1/4}\, t^{(\alpha+2)/4}$

Froude number constant

[Formulae must be dimensionally correct]

with constants only really determinable by numerical solution of equations or experiment, with good agreement of functional forms.

simplest analysis of approximate box model 6.3



$\alpha = 0$; constant V

$$\dot{x}_N = Fr(g'h)^{1/2}$$

$$x_N h = A_0$$

$$x_N = \left(\tfrac{3}{2} Fr\right)^{2/3} (g'A_0)^{1/3} t^{2/3}$$

(no entrainment)

[Shallow water theory also possible, B,H[2] & L, 1993]

2. Low Reynolds number $Re = Vh/\nu << 1$ currents are very different. Inertia is no longer relevant.

viscous force

$$F_v = \int \mu \;\; ^2 u \, dx \, dy \, dz$$

$$\sim \mu U L w / h$$

$$\sim \mu q^{-1} L^3 w t^{-\alpha-1}$$

$$F_v \sim \mu U R^2 / h$$

$$\sim \mu Q^{-1} R^5 t^{-\alpha-1}$$

buoyancy - viscous balance $F_v \sim F_g$

$$L \sim \left(g' q^3/\nu\right)^{1/5} t^{(3\alpha+1)/5} \qquad R \sim \left(g' Q^3/\nu\right)^{1/8} t^{(3\alpha+1)/8}$$

Under what conditions are these balances valid? 6.4

$$F_i / F_v \sim \left(q^4 g'^2 \nu^3\right)^{1/5} t^{(4\alpha-7)/5} \qquad \sim (Q/g'\nu)^{1/2} t^{(\alpha-3)/2}$$

which allows for the definition of a transition time

$$t_1 = \left(q^4/g'^2 \nu^3\right)^{1/(7-4\alpha)} \qquad t_1 = (Q/g'\nu) t^{1/(3-\alpha)}$$

These expressions have a singularity at

$$\alpha = \alpha_c = 7/4 \qquad \alpha = \alpha_c = 3$$

For $\alpha < \alpha_c$ $t << t_1$ inertial } current
  $t >> t_1$ viscous }

$\alpha = \alpha_c$ $J = \nu^3 g'^2 / q^4, \nu g'/Q$ $<< 1$ inertial t
  $>> 1$ viscous t

$\alpha > \alpha_c$ $t << t_1$ viscous } current
  $t >> t_1$ inertial }

$\alpha < \alpha_c$ inertial then viscous

$\alpha > \alpha_c$ viscous then inertial

3. $Re << 1$ $\mathbf{q}$ $\mathbf{q} \approx \mathbf{0}$ 6.5

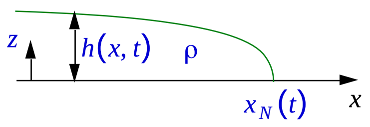
$$0 = - \quad p + \mu \quad ^2 \mathbf{q}$$

(linear equation, but boundary conditions may not be)

In the *flow of thin films* - lava domes, lubrication bearings - analytical simplifications can be made which take into account that $\partial_x << \partial_z$ and

$\mathbf{u} \approx [u(z; x), 0, 0]$.

As an example, consider a thin dimensional current propagating on a rigid horizontal boundary - lava flow, honey on toast, oil on ground etc.



$p = p_0$ on unknown surface $z = h(x, t)$

$$p = p_0 + \rho g (h - z)$$

hydrostatic

(H², 1982)

$$-\frac{\partial p}{\partial x} = -\rho g \frac{\partial h}{\partial x}$$

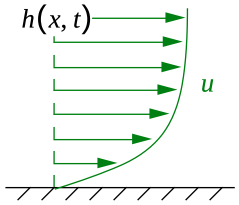
Current driven by pressure gradient as a result of slope of free surface. 6.6

[If density of outer fluid is $\rho - \Delta\rho$, g' replaces g.]

0 because thin

(x - momentum) $0 = -\rho g \frac{\partial h}{\partial x}(x, t) + \mu u_{xx} + \mu u_{zz}$

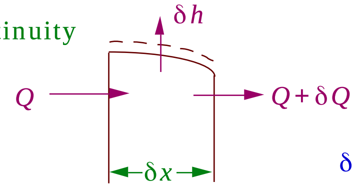
(boundary conditions) $u = 0 \ (z = 0)$ $\frac{\partial u}{\partial z} = 0 \ (z = h)$



$$u(x, z, t) = -\frac{g}{2\nu}\frac{\partial h}{\partial x} z(2h - z)$$

$$Q = \int_0^h u \, dz = -\tfrac{1}{3}\frac{g}{\nu} h^3 \frac{\partial h}{\partial x} \qquad (\equiv L^2 T^{-1})$$

local continuity



$$\delta h \delta x + \delta Q \delta t = 0$$

6.7

i.e. $\frac{\partial h}{\partial t}+\frac{\partial Q}{\partial x}=0$

and hence $\frac{\partial h}{\partial t}-\frac{1}{3}\frac{g}{\nu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right)=0$

a nonlinear diffusion equation

global continuity $\int_0^{x_N(t)} h(x,t)\,dx = \text{Volume} \quad = At^{\alpha}$ say

Solution either by numerical integration for given initial conditions (difficult) or by trying for a solution in one similarity variable [c.f. $x/(\kappa t)^{\frac{1}{2}}$] to which all solutions tend.

H^2 method to determine similarity solutions

$$\beta=\frac{1}{3}\frac{g}{\nu} \qquad \frac{h}{t}\sim\beta\frac{h^4}{x^2} \qquad h^3\sim x^2/\beta t$$

same size as, and definitely

same dimensions as

6.8

$hx\sim V$ (assume constant; $\alpha = 0$) $h\sim V/x$

$\frac{V^3}{x^3}\sim\frac{x^2}{\beta t}$ or $\frac{x^5}{\beta V^3 t}\sim 1$ $\quad (\beta V^3)^{-1/5}xt^{-1/5}\sim 1$

suggests $\eta=(\beta V^3)^{-1/5}xt^{-1/5}$ is a suitable similarity variable

and also $h\sim\frac{V}{x}\sim\left(\frac{V^2}{\beta}\right)^{1/5}t^{-1/5}$

Thus $h(x,t)=\eta_N^{2/3}\left(V^2/\beta\right)^{1/5}t^{-1/5}\phi(\eta/\eta_N\equiv y)$

where η_N is the value of η at the nose

$\partial_t=-\frac{1}{5}\frac{\eta}{t}d_\eta \qquad \partial_x=\frac{\eta}{x}d_\eta$

with $\left(\phi^3\phi'\right)'+\frac{1}{5}y\phi'+\frac{1}{5}\phi=0 \qquad {}'=\frac{d}{dy}$

$\eta_N=\left(\int_0^1\phi\,dy\right)^{-3/5}$

1 b.c. $\phi=0$ $(y=1)$ suffices!!

exact solution by integration $\phi(y)=\left(\frac{3}{10}\right)^{1/3}\left(1-y^2\right)^{1/3}$ $\quad \eta_N=1.411$

(in terms of Γ functions)

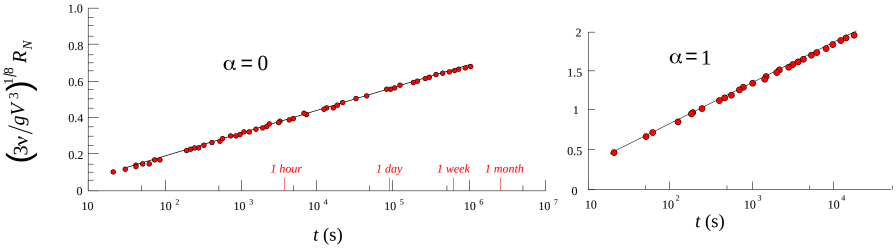
If $\alpha \neq 0$ similarity theory works only if input at origin (otherwise a length scale is introduced) with similarity variable 6.9

$$\eta = \left(\beta V^3\right)^{-1/5} x\, t^{-(3\alpha+1)/5}$$

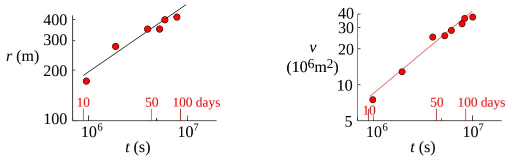
which shows immediately that $x_N = \eta_N \left(\beta V^3\right)^{1/5} t^{(3\alpha+1)/5}$

if the flow is axisymmetric $R_N(t) = \xi_N \left(\beta V^3\right)^{1/8} t^{(3\alpha+1)/8}$

Experimental confirmation



The 1979 eruption of the Soufriere of St. Vincent (H² et al 1982) 6.10



The normalized radius and evaluated volume of the 1979 dome of the Soufriere of St. Vincent, and the lines of bets fit to the data.

$r = 90\ t^{0.39}$ $\qquad v = 933\ t^{0.66}$

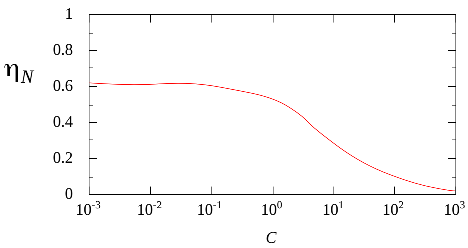
$\alpha = 0.66 \qquad \dfrac{3\alpha + 1}{8} = 0.37$ ✓ $\qquad \nu = 6 \times 10^7\ \ m^2 s^{-1}$

If compressibility of the flow is important because of gas pockets, the density will increase as a function of pressure, e.g. $\rho = \rho_0 \left\{1 - B\left(p - p_0\right)\right\}$

The importance of compressibility in a flow of constant influx, Q, is dependent upon the value of

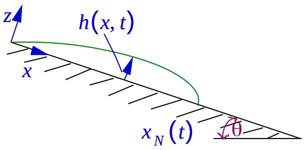
$$C = \rho_0 B g \left(\mu Q / \rho_0^2 g\right)^{1/4}$$

$$r = \eta_N \left(g Q^3 / \nu \rho_0^3\right)^{1/8} t^{1/2}$$



Is a two-dimensional flow of a thin film down a slope possible? 6.11

(H[2], 1982)



Assume it is

(x - momentum) $0 = -\frac{1}{\rho} p_x + g \sin\theta + \nu u_{zz}$

(z - momentum) $0 = -\frac{1}{\rho} p_z - g \cos\theta$

$$p(x, z, t) = -\rho g z \cos\theta + f(x, t)$$

with $f(x,t)$ such that $p = p_0$ on $z = h(x,t)$, i.e.

$p = p_0 + \rho g (h - z)\cos\theta$ $\qquad \nu u_{zz} = g h_x c_\theta - g \sin\theta$

$u(x, 0) = 0 \qquad u_z(x, h) = 0$

unless $\alpha \approx 0$, because $h_x << 1$, $h_x \cos\theta << \sin\theta$

$$u = \frac{g \sin\theta}{2\nu} z(2h - z) \quad \text{with} \quad Q = \tfrac{1}{3}\frac{g}{\nu} h^3 \sin\theta$$

$$h_t + Q_x = 0 \qquad h_t + \frac{g \sin\theta}{\nu} h^2 h_x = 0$$

A first order nonlinear equation which shows that h is constant along characteristics given by $\frac{dx}{dt} = \frac{g \sin\theta}{\nu} h^2$

The solution is thus given by $h(x,t) = F\left(x - \frac{g s_\alpha}{\nu} h^2 t\right)$ 6.12

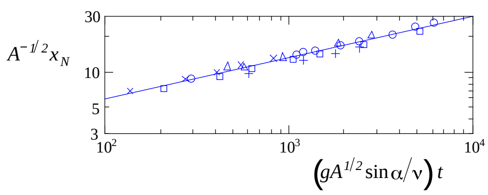
where $h(x, 0) = F(x)$

Larger values of h travel faster (h^2) with initial blob spreading so that back thins and front steepens

As $t \to \infty$ the solution must behave like $h = (\nu / g s_\alpha)^{1/2} x^{1/2} t^{-1/2}$ independent of F. This is also the similarity solution [Prove this!]

Considered along with $\int_0^{x_N(t)} h(x,t)\, dx = A,$

we obtain $x_N = \left(\tfrac{9}{4} A^2 g s_\alpha / \nu\right)^{1/3} t^{1/3}$



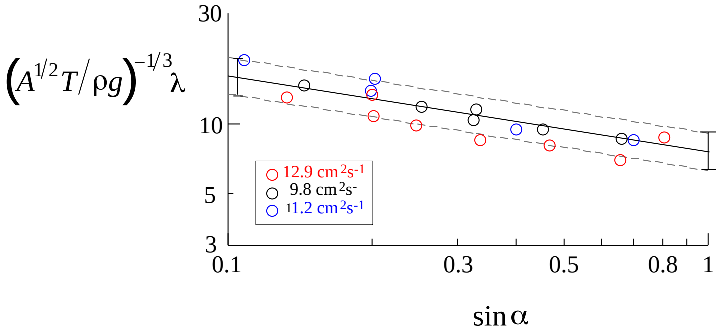
The length of a two-dimensional current down a slope, normalized with respect to $A^{1/2}$, as a function of suitably non-dimensionalized time. The straight line is the theoretical prediction and the experimental points are for six typical runs.

6.13

Incorporating the effects of surface tension (coefficient T) indicates that after a distance x_c $A^{1/2}$ an instability occurs with a wavelength

$$\lambda = 7.5\left(A^{1/2}T/\rho g s_\alpha\right)^{1/3}$$

independent of ν (which only gives timescale) !



The wavelength of the instability at the front, normalized with respect to $\left(A^{1/2}T/\rho g\right)^{1/3}$, as a function of the slope angle α , at three different viscosities.

6.14

MAIN CONCEPTS

- Gravity currents propagate with buoyancy forces balanced by either inertial forces, for $Re >> 1$, or viscous forces, for $Re << 1$.
- Force balances or simple box models for the evolution lead to good predictions.
- Lubrication theory for viscous currents yields very accurate descriptions.
- Similarity theory is often useful in advection / diffusion problems.

6.15

Lecture 6. Gravity Currents and Lava Domes

Bonnecaze, R.T., Huppert, H.E. and Lister, J.R., 1993 Particle-driven gravity currents, *J. Fluid Mech.*, **250**, 339-369.

Huppert, H.E., 1982 The flow and instability of viscous gravity currents down a slope, *Nature*, **300**, 427-429 with front cover photograph.

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Huppert, H.E., Shepherd, J.B., Sigurdsson, H. and Sparks, R.S.J.,1982 On lava dome growth, with applications to the 1979 lava extrusion of Soufriere, St. Vincent, *J. Volcanol. and Geotherm. Res.*, **14**, 199-222.