Singular Poisson brackets

For the pefect fluid equations we have two alternative formulations of Hamiltonian dynamics,

$$\frac{dF}{dt} = \{F, H\}$$

<u>Lagrangian formulation</u>

$$\{A,B\} = \iiint d\mathbf{a} \left(\frac{\delta A}{\delta \mathbf{x}(\mathbf{a})} \frac{\delta B}{\delta \mathbf{v}(\mathbf{a})} - \frac{\delta A}{\delta \mathbf{v}(\mathbf{a})} \frac{\delta B}{\delta \mathbf{x}(\mathbf{a})} \right)$$

$$H = \iiint d\mathbf{a} \left\{ \tfrac{1}{2} \mathbf{v}(\mathbf{a})^2 + E\left(\frac{\partial(\mathbf{a})}{\partial(\mathbf{x})}, S(\mathbf{a}) \right) + \Phi(\mathbf{x}(\mathbf{a})) \right\}$$

involves the 6 variables

$$\mathbf{v}(\mathbf{a}),\ \mathbf{x}(\mathbf{a})$$

<u>Eulerian formulation</u>

$$\{A,B\} = \iiint d\mathbf{x}\ [\ \nabla\left(\frac{\delta A}{\delta \rho}\right) \cdot \frac{\delta B}{\delta \mathbf{v}} - \nabla\left(\frac{\delta B}{\delta \rho}\right) \cdot \frac{\delta A}{\delta \mathbf{v}} + \frac{\nabla \times \mathbf{v}}{\rho} \cdot \frac{\delta A}{\delta \mathbf{v}} \times \frac{\delta B}{\delta \mathbf{v}}$$
$$+ \frac{\nabla S}{\rho} \cdot \left(\frac{\delta A}{\delta \mathbf{v}} \frac{\delta B}{\delta S} - \frac{\delta B}{\delta \mathbf{v}} \frac{\delta A}{\delta S} \right)\ ]$$

$$H = \iiint d\mathbf{x}\ \ \rho(\mathbf{x})\ \left\{ \tfrac{1}{2} \mathbf{v}(\mathbf{x})^2 + E\left(\frac{1}{\rho(\mathbf{x})}, S(\mathbf{x}) \right) + \Phi(\mathbf{x}) \right\}$$

involves the 5 variables

$$\mathbf{v}(\mathbf{x}),\ \rho(\mathbf{x}),\ S(\mathbf{x})$$

Thus the transformation from the Lagrangian to the Eulerian formulation is *projective*.

Because of this, the Eulerian bracket is singular: There exists functionals C for which

$$\{A,C\} = 0 \qquad \text{for every } A$$

In fact

$$C = \iiint d\mathbf{x} \ \ \rho \, F(S,q)$$

where F is an arbitrary function, and

$$q = \frac{(\nabla \times \mathbf{v}) \cdot \nabla S}{\rho}$$

is the potential vorticity.

C is called a *Casimir*.

Summary for the case of discrete variables

$$\frac{dz^i}{dt} = J^{ij}\frac{\partial H}{\partial z^j}, \qquad i = 1,\ldots,2N$$

Let *J* be antisymmetric and obey the Jacobi identity.

If *J* is <u>nonsingular</u>, then it can always be brought into the canonical form (Darboux)

$$J = \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix}$$

If *J* is <u>singular</u> with corank *K*=2*M*, then it can be brought into the form

$$\left[J^{\alpha\beta}\right] = \begin{bmatrix} 0_M & I_M & 0 \\ -I_M & 0_M & 0 \\ 0 & 0 & 0_{n-2M} \end{bmatrix}$$

Even in the case of discrete variables, it can be very difficult to attain these ideal forms. The most useful result is that the *K* nullvectors of a singular *J* are the gradients of *K* scalars—the Casimirs.

$$J^{ij}\frac{\partial C^{(k)}}{\partial z^j} = 0, \qquad k = 1,\ldots,K$$

Summary for the case of continuous variables

$$\frac{dF}{dt} = \{F,H\}$$

Let the bracket be antisymmetric

$$\{A,B\} = -\{B,A\}$$

and obey the Jacobi identity

$$\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$$

If the bracket is singular, there exist a set of Casimirs C such that

$$\{C,B\} = 0$$

for any B. In particular,

$$\{C,H\} = 0$$

The Casimir is conserved, for any Hamiltonian.

Lagrangian brackets are nonsingular.
Eulerian brackets are singular.

Example: Find the Casimirs for the quasigeostrophic bracket

The QG equation

$$\frac{dA}{dt} = \iint dx\,dy\;\; q\; J\left(\frac{\delta A}{\delta \zeta}, \frac{\delta H}{\delta \zeta}\right)$$

involves the bracket

$$\{A,B\} = \iint dx\,dy\;\; q\; J\left(\frac{\delta A}{\delta \zeta}, \frac{\delta B}{\delta \zeta}\right)$$

where

$$q = \zeta + h$$

is the potential vorticity.

If C is a Casimir,

$$\{C,B\} = \iint dx\,dy\;\; q\; J\left(\frac{\delta C}{\delta \zeta}, \frac{\delta B}{\delta \zeta}\right) = -\iint dx\,dy\;\; J\left(\frac{\delta C}{\delta \zeta}, q\right)\frac{\delta B}{\delta \zeta} = 0$$

for *any* B. Therefore, it must be true that

$$J\left(\frac{\delta C}{\delta \zeta}, q\right) = 0$$

which implies

$$\frac{\delta C}{\delta \zeta} = F'(q)$$

for some function F. It follows that

$$C = \iint dx\,dy\;\; F(q)$$

It is often difficult to compute the Casimirs, but they are usually some combination of the potential vorticity and other conserved scalars.

Example: the shallow water equations

In Eulerian variables, the dynamics is

$$\frac{dF}{dt} = \{F,H\}$$

where

$$H = \iint d\mathbf{x}\left(\tfrac{1}{2}hu^2 + \tfrac{1}{2}hv^2 + \tfrac{1}{2}gh^2\right)$$

is the Hamiltonian with **u**=(u,v), and

$$\{A,B\} = \iint d\mathbf{x}\left(q\frac{\delta(A,B)}{\delta(u,v)} - \frac{\delta A}{\delta \mathbf{u}}\cdot\nabla\frac{\delta B}{\delta h} + \frac{\delta B}{\delta \mathbf{u}}\cdot\nabla\frac{\delta A}{\delta h}\right)$$

is the Poisson bracket, with

$$\frac{\delta(A,B)}{\delta(u,v)} = \frac{\delta A}{\delta u}\frac{\delta B}{\delta v} - \frac{\delta B}{\delta u}\frac{\delta A}{\delta v}$$

The Casimirs are

$$C = \iint d\mathbf{x}\ hG(q)$$

where

$$q = \left(v_x - u_y\right)h^{-1}$$

(Prove this.)

Poisson brackets allow a Hamiltonian formulation of fluid mechanics that involves only the conventional Eulerian variables.

No particle labels or Clebsch potentials are required.

The Eulerian formulation is closed, but incomplete: It does not keep track of where fluid particles go.

The Eulerian bracket is singular, and its “null vectors” are the gradients of Casimirs. The Casimirs are conserved quantities related to the particle-relabelling symmetry. This symmetry, which is responsible for potential vorticity conservation, is also responsible for the existence of a closed Eulerian dynamics.

What practical use are the Eulerian brackets?

Few applications depend on the precise form of the bracket.

Many applications make use of the Casimirs.

Discrete system with a singular Poisson bracket

$$\frac{dz^i}{dt} = J^{ij}\frac{\partial H}{\partial z^j} \qquad \{z^i(t),\ i=1 \text{ to } n\}$$

Since J is singular,

$$J^{ij}\frac{\partial C^{(k)}}{\partial z^j} = 0$$

Let

$$z_0 = \left(z_0^1, z_0^2, \ldots, z_0^n\right)$$

be a fixed point, i.e.

$$\frac{dz^i}{dt} = J^{ij}\frac{\partial H}{\partial z^j} = 0 \quad \text{at} \quad z = z_0$$

If J were nonsingular this would imply

$$\frac{\partial H}{\partial z^j} = 0 \quad \text{at} \quad z = z_0$$

Since J is singular,

$$\frac{\partial H}{\partial z^j} = -\sum_{k=1}^{K}\lambda_k\frac{\partial C^{(k)}}{\partial z^j} \quad \text{at} \quad z = z_0$$

The constants $\{\lambda_k\}$ depend on the fixed point. Thus every fixed point is a stationary point of

$$I(z) \equiv H(z) + \sum_k \lambda_k C^{(k)}(z)$$

for some $\{\lambda_k\}$.

$$\frac{\partial I}{\partial z^j} = 0 \quad \text{at} \quad z = z_0$$

where

$$I(z) \equiv H(z) + \sum_k \lambda_k C^{(k)}(z)$$

Let

$$z = z_0 + \Delta z \quad \text{where} \quad \Delta z = \left(\Delta z^1, \Delta z^2, \ldots, \Delta z^n\right)$$

is disturbance amplitude.

Define the *pseudoenergy*

$$\Delta I(\Delta z; z_0) \equiv I(z_0 + \Delta z) - I(z_0)$$

Pseudoenergy is a conserved quantity that vanishes at the fixed point.

A Taylor-series expansion yields

$$\Delta I(\Delta z; z_0) = H_L + O\left((\Delta z)^3\right) = \tfrac{1}{2}\frac{\partial^2 I}{\partial z^i \partial z^j}(z_0)\,\Delta z^i \Delta z^j + O\left((\Delta z)^3\right)$$

where

$$H_L = \tfrac{1}{2}\frac{\partial^2 I}{\partial z^i \partial z^j}(z_0)\,\Delta z^i \Delta z^j$$

Thus the pseudoenergy is *second order* in the disturbance-amplitude Δz.

This is not true of the energy!

Pseudoenergy

$$\Delta I(\Delta z; z_0) = H_L + O\left((\Delta z)^3\right) = \tfrac{1}{2}\frac{\partial^2 I}{\partial z^i \partial z^j}(z_0)\,\Delta z^i \Delta z^j + O\left((\Delta z)^3\right)$$

H_L is the Hamiltonian for the linearized dynamics.

Proof:

$$\frac{d}{dt}\left(z_0{}^i + \Delta z^i\right) = J^{ij}(z_0 + \Delta z)\,\frac{\partial H}{\partial z^j}(z_0 + \Delta z)$$

$$= \left[J^{ij}(z_0) + \frac{\partial J^{ij}}{\partial z^m}(z_0)\Delta z^m\right]\left[\frac{\partial H}{\partial z^j}(z_0) + \frac{\partial^2 H}{\partial z^j \partial z^m}(z_0)\,\Delta z^m\right] + O\left((\Delta z)^2\right)$$

$$= \left[\frac{\partial J^{ij}}{\partial z^m}\frac{\partial H}{\partial z^j} + J^{ij}\frac{\partial^2 H}{\partial z^j \partial z^m}\right]_0 \Delta z^m + O\left((\Delta z)^2\right)$$

But

$$\frac{\partial J^{ij}}{\partial z^m}\frac{\partial H}{\partial z^j} = -\frac{\partial J^{ij}}{\partial z^m}\sum_k \lambda_k \frac{\partial C^{(k)}}{\partial z^j} = J^{ij}\sum_k \lambda_k \frac{\partial^2 C^{(k)}}{\partial z^m \partial z^j}$$

Thus the linear dynamics is

$$\frac{d}{dt}\Delta z^i = J^{ij}(z_0)\frac{\partial H_L}{\partial \Delta z^j}$$

However H_L is not an approximation to H.

Stability

The linear dynamics is stable if

$$H_L = \tfrac{1}{2} \frac{\partial^2 I}{\partial z^i \partial z^j}(z_0)\, \Delta z^i \Delta z^j$$

has definite sign.

For finite degrees of freedom, this means that the nonlinear dynamics is stable too.

For infinite degrees of freedom, H_L offers no help, but the form of the pseudoenergy may suggest a bound.

Example: quasigeostrophic flow

$$\frac{\partial}{\partial t} \nabla^2 \psi + J\left(\psi, \nabla^2 \psi + h(x,y)\right) = 0$$

Casimir

$$C = \iint dx\, dy\; F(q), \qquad q = \nabla^2 \psi + h$$

The steady solution $\psi_0(\mathbf{x})$ is a stationary point of

$$I[\psi(\mathbf{x})] = H + C = \iint d\mathbf{x} \left\{ \tfrac{1}{2} \nabla\psi \cdot \nabla\psi + F(q) \right\}$$

Indeed

$$\delta I = \iint d\mathbf{x} \left\{ \left(-\psi + F'(q)\right) \delta\zeta \right\} = 0$$

implies

$$\psi_0 = F'\left(\nabla^2\psi_0 + h\right) \equiv \Psi(q_0)$$

Ψ is analogous to $\{\lambda_k\}$. Let

$$\psi(\mathbf{x},t) = \psi_0(\mathbf{x}) + \psi'(\mathbf{x},t)$$

ψ_0 is analogous to z_0. ψ' is analogous to Δz.

The pseudoenergy is

$$\begin{aligned}
&\Delta I = I[\psi_0 + \psi'] - I[\psi_0] \\
&= \iint d\mathbf{x}\left\{\tfrac{1}{2}\nabla(\psi_0 + \psi')\cdot\nabla(\psi_0 + \psi') - \tfrac{1}{2}\nabla\psi_0\cdot\nabla\psi_0 + F(q_0 + q') - F(q_0)\right\} \\
&= H_L + O\left((\psi')^3\right)
\end{aligned}$$

where

$$H_L = \iint d\mathbf{x}\left\{\tfrac{1}{2}\nabla\psi'\cdot\nabla\psi' + \tfrac{1}{2}F''(q_0)\left(\nabla^2\psi'\right)^2\right\}$$

which is indeed conserved by the linear dynamics,

$$\frac{\partial\nabla^2\psi'}{\partial t} + J\left(\psi_0, \nabla^2\psi'\right) + J\left(\psi', \nabla^2\psi_0 + h\right) = 0$$

According to *linear dynamics* the steady state is stable if

$$F''(q_0) = \frac{d\Psi}{dq_0} > 0 \quad \text{(Arnol'd's theorem)}$$

or

$$\frac{d\Psi}{dq_0} < -\frac{1}{{k_{\min}}^2} \quad \text{(Arnol'd's second theorem)}$$

Nonlinear stability

Manipulate the pseudomomentum into a simpler form:

$$\begin{aligned}
\Delta I &= I[\psi_0 + \psi'] - I[\psi_0] \\
&= \iint d\mathbf{x}\left\{\tfrac{1}{2}\nabla(\psi_0+\psi')\cdot\nabla(\psi_0+\psi') - \tfrac{1}{2}\nabla\psi_0\cdot\nabla\psi_0 + F(q_0+q') - F(q_0)\right\} \\
&= \iint d\mathbf{x}\left\{\tfrac{1}{2}\nabla\psi'\cdot\nabla\psi' + \nabla\psi_0\cdot\nabla\psi' + F(q_0+q') - F(q_0)\right\} \\
&= \iint d\mathbf{x}\left\{\tfrac{1}{2}\nabla\psi'\cdot\nabla\psi' - \Psi(q_0)\nabla^2\psi' + \int_{q_0}^{q_0+q'}\Psi(q)\,dq\right\} \\
&= \iint d\mathbf{x}\left\{\tfrac{1}{2}\nabla\psi'\cdot\nabla\psi' - \Psi(q_0)\,q' + \int_0^{q'}\Psi(q_0+\tilde{q})\,d\tilde{q}\right\} \\
&= \iint d\mathbf{x}\left\{\tfrac{1}{2}\nabla\psi'\cdot\nabla\psi' + \int_0^{q'}\left\{\Psi(q_0+\tilde{q}) - \Psi(q_0)\right\}d\tilde{q}\right\}
\end{aligned}$$

Assume

$$0 < c_1 < \frac{d\Psi}{dq} < c_2$$

Then

$$\iint d\mathbf{x}\left\{\nabla\psi'\cdot\nabla\psi' + c_1(q')^2\right\} \le 2\,\Delta I(t) = 2\Delta I(0) \le \iint d\mathbf{x}\left\{\nabla\psi_0'\cdot\nabla\psi_0' + c_2(q_0')^2\right\}$$

Define

$$\|\psi'\|^2 \equiv \iint d\mathbf{x}\left\{\nabla\psi'\cdot\nabla\psi' + c_1(q')^2\right\}$$

Then

$$\|\psi'(t)\|^2 < \frac{c_2}{c_1}\|\psi'(0)\|^2$$

This approach to finite-amplitude stability seems to work well only for systems with quadratic conservation laws, not e.g. for the shallow water equations. (Ripa 1983, 1992; Shepherd 2002).

Quite apart from **stability** the concept of pseudoenergy (and its generalizations) provides an important unifying framework for all kinds of conservation laws.

Example: Available internal energy

The general perfect-fluid equations conserve the energy

$$H = \iiint d\mathbf{x} \left\{ \tfrac{1}{2} \rho\, \mathbf{v} \cdot \mathbf{v} + \rho\, E\left(\frac{1}{\rho}, S \right) + \rho\, \Phi(\mathbf{x}) \right\}$$

and the Casimirs

$$C = \iiint d\mathbf{x}\, \rho\, F(S, q)$$

Functional derivatives of the Hamiltonian:

$$\frac{\delta H}{\delta \rho} = \tfrac{1}{2} \mathbf{v} \cdot \mathbf{v} + E + \frac{p}{\rho}, \qquad p \equiv -\frac{\partial E(\alpha, S)}{\partial \alpha}$$

$$\frac{\delta H}{\delta \mathbf{v}} = \rho \mathbf{v}$$

$$\frac{\delta H}{\delta S} = \rho T, \qquad T \equiv \frac{\partial E(\alpha, S)}{\partial S}$$

Functional derivatives of the Casimir:

$$\frac{\delta C}{\delta \rho} = F - q\, F_q$$

$$\frac{\delta C}{\delta \mathbf{v}} = \nabla \times \left(F_q \nabla S \right)$$

$$\frac{\delta C}{\delta S} = \rho\, F_S - (\nabla \times \mathbf{v}) \cdot \nabla F_q$$

Choose F such that

$$\frac{\delta C}{\delta \rho} = -\frac{\delta H}{\delta \rho}, \qquad \frac{\delta C}{\delta \mathbf{v}} = -\frac{\delta H}{\delta \mathbf{v}}, \qquad \frac{\delta C}{\delta S} = -\frac{\delta H}{\delta S}$$

for the chosen stationary state. If the stationary state is a state of rest, then these equations have the solution

$$F(S) = -E\left(\frac{1}{\rho_0}, S\right) + \frac{1}{\rho_0} E_\alpha\left(\frac{1}{\rho_0}, S_0\right) \equiv -E\left(\frac{1}{\rho_0}, S\right) - \frac{p_0}{\rho_0}$$

Therefore

$$I = H + C = \iiint d\mathbf{x}\left\{\tfrac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \rho E\left(\frac{1}{\rho}, S\right) - \rho E\left(\frac{1}{\rho_0}, S\right) - \frac{\rho}{\rho_0} p_0\right\}$$

and

$$\Delta I = I\left[\rho(\mathbf{x}), \mathbf{v}(\mathbf{x}), S(\mathbf{x})\right] - I\left[\rho_0(\mathbf{x}), \mathbf{v}_0(\mathbf{x}), S_0(\mathbf{x})\right]$$

$$= \iiint d\mathbf{x}\left\{\tfrac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \rho E\left(\frac{1}{\rho}, S\right) - \rho E\left(\frac{1}{\rho_0}, S\right) - \frac{(\rho - \rho_0)}{\rho_0} p_0\right\}$$

Suppose the fluid is homentropic. Let

$$\rho'(\mathbf{x}, t) = \rho(\mathbf{x}, t) - \rho_0(\mathbf{x}), \qquad \mathbf{v}'(\mathbf{x}, t) \equiv \mathbf{v}(\mathbf{x}, t)$$

Then

$$\Delta I \approx \iiint d\mathbf{x}\left\{\tfrac{1}{2}\rho_0\mathbf{v}'\cdot\mathbf{v}' + \tfrac{1}{2}\frac{1}{\rho_0{}^3} E_{\alpha\alpha}\left(\frac{1}{\rho_0}\right)(\rho')^2\right\}$$

$$= \iiint d\mathbf{x}\left\{\tfrac{1}{2}\rho_0\mathbf{v}'\cdot\mathbf{v}' + \tfrac{1}{2}\frac{c^2}{\rho_0}(\rho')^2\right\}$$

The last term is *available internal energy.*

Generalizations

General statement:

$$\{z, G\} = J^{ij} \frac{\partial G}{\partial z^j} = 0 \qquad \text{at} \quad z = z_0$$

If G=H the mean flow has time-translation symmetry; this is the case we have considered so far.

Suppose G=M, where M is the momentum in a particular direction. That is, suppose that the mean flow is invariant in the direction corresponding to M.

Then proceeding in exactly the same way as before, we conclude that

$$I(z) \equiv M(z) + \sum_k \lambda_k C^{(k)}(z)$$

is stationary at the mean state, and therefore

$$\Delta I = I(z_0 + \Delta z) - I(z_0)$$

called *pseudomomentum,* is a second-order conserved quantity that vanishes in the mean state.

If the mean state is both steady and invariant to a space translation, then we may use

$$I(z) \equiv H(z) + \lambda_0 G(z) + \sum_k \lambda_k C^{(k)}(z)$$

for constructing stability bounds, and measures of available energy.