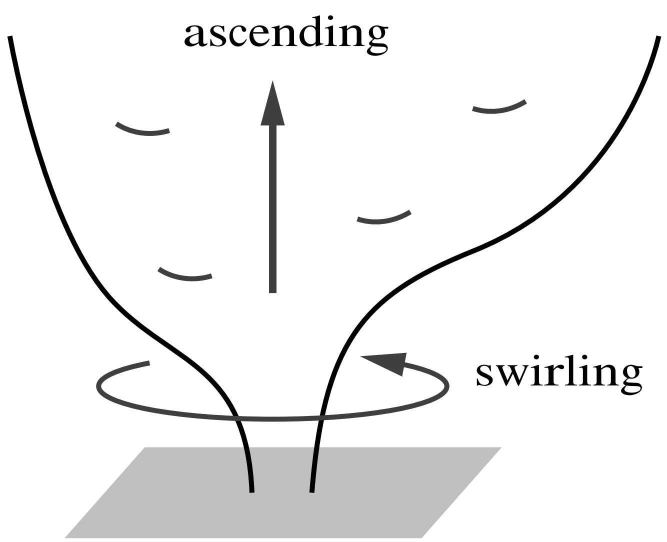
GFD Seminar
Shikotsu-Lake, 15 March 2025

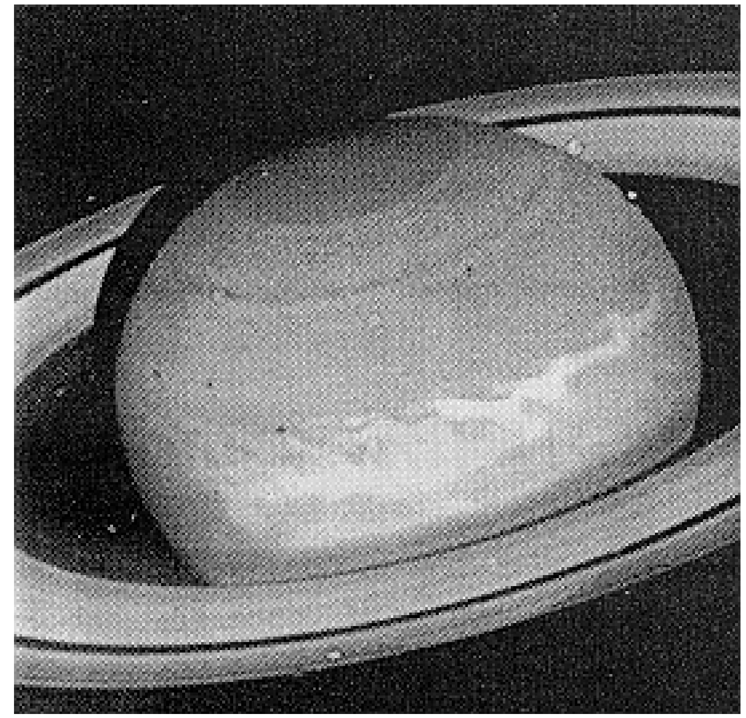
Modelling turbulence based on response-function and multiple-scale formulation

Nobumitsu YOKOI
Institute of Industrial Science (IIS), Univ. of Tokyo

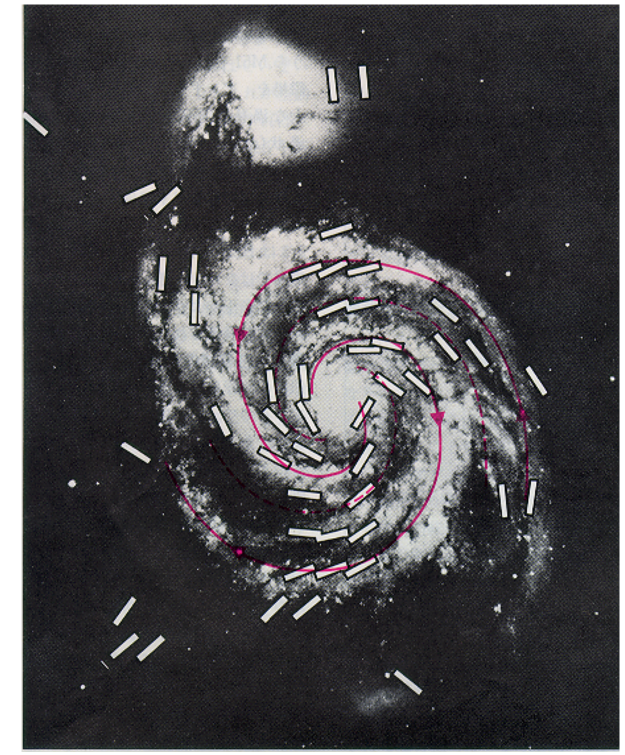
Large-scale structures in Turbulence



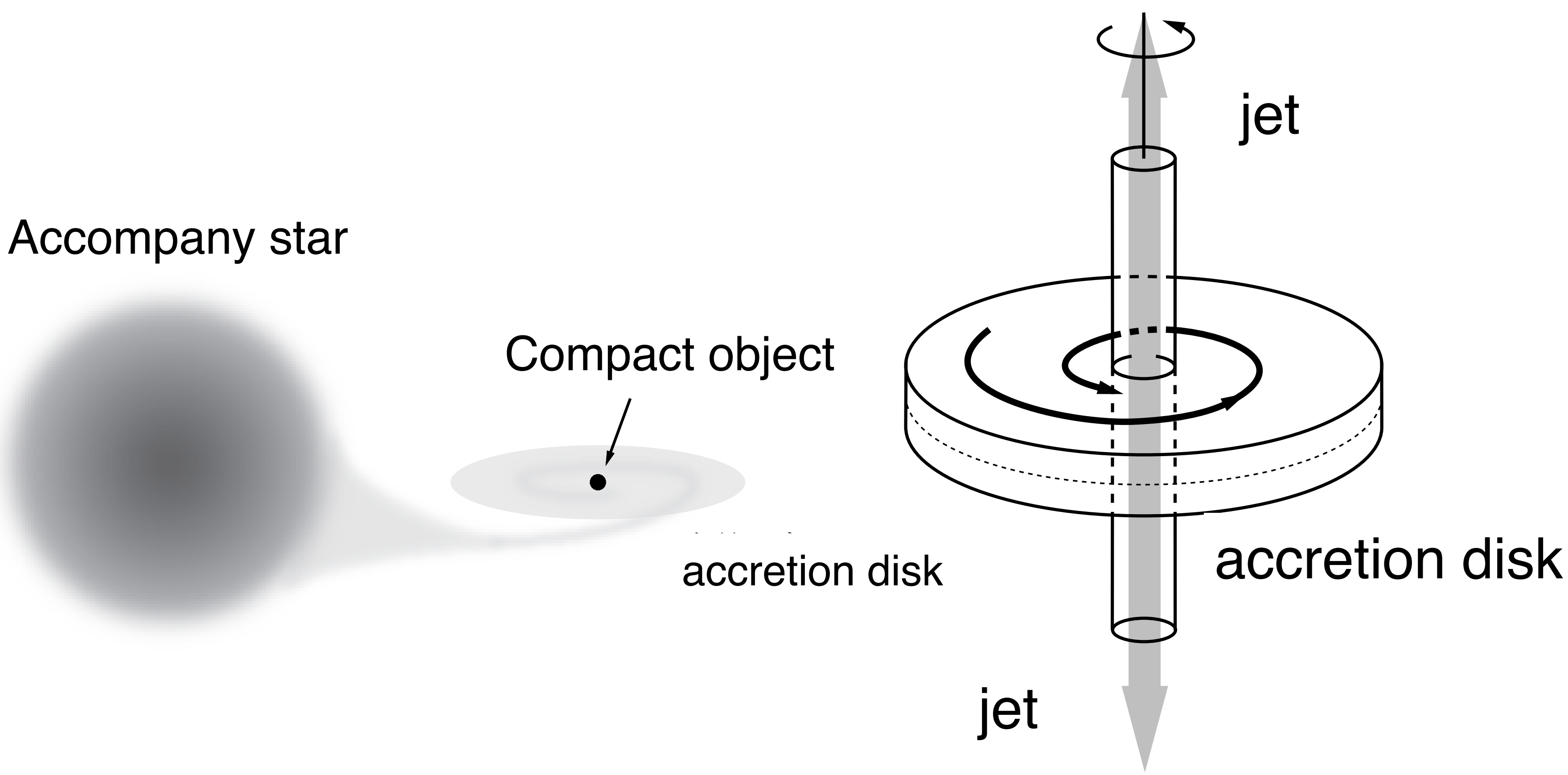
Typhoon, tornado



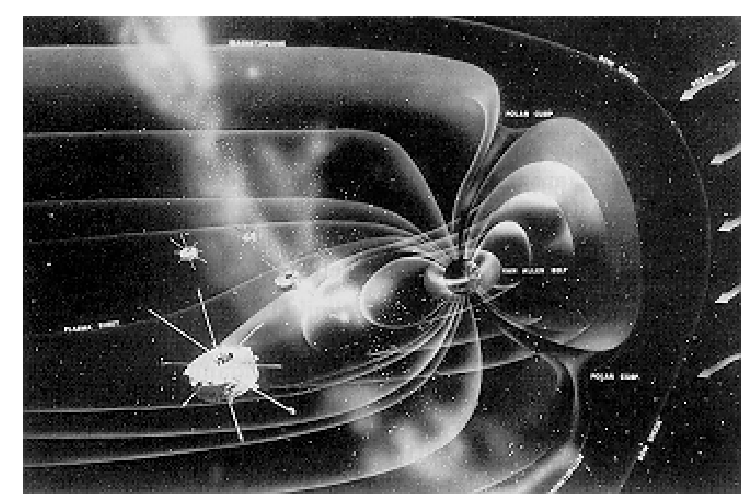
White spot on Saturn



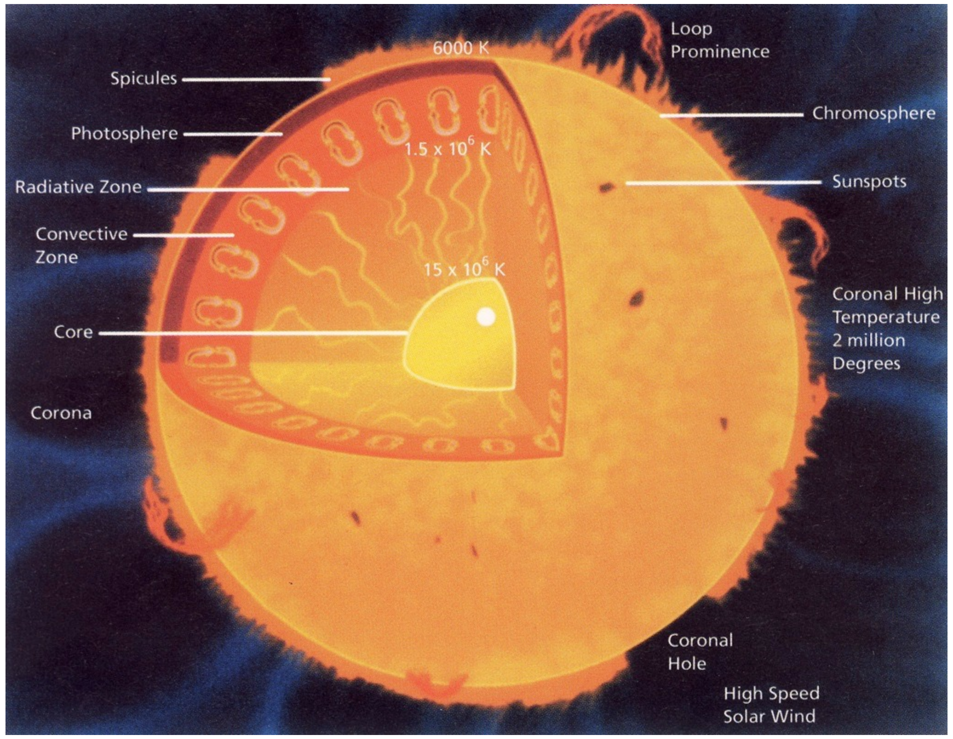
Magnetic field configuration
in galaxy (M51)



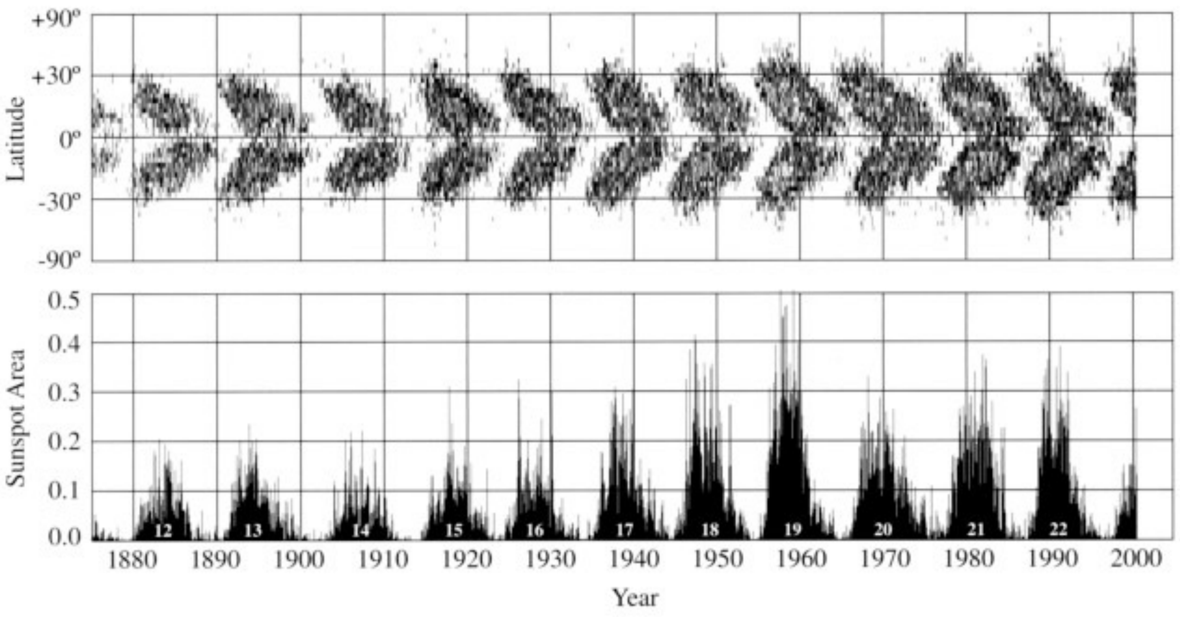
Accretion disk and its jet



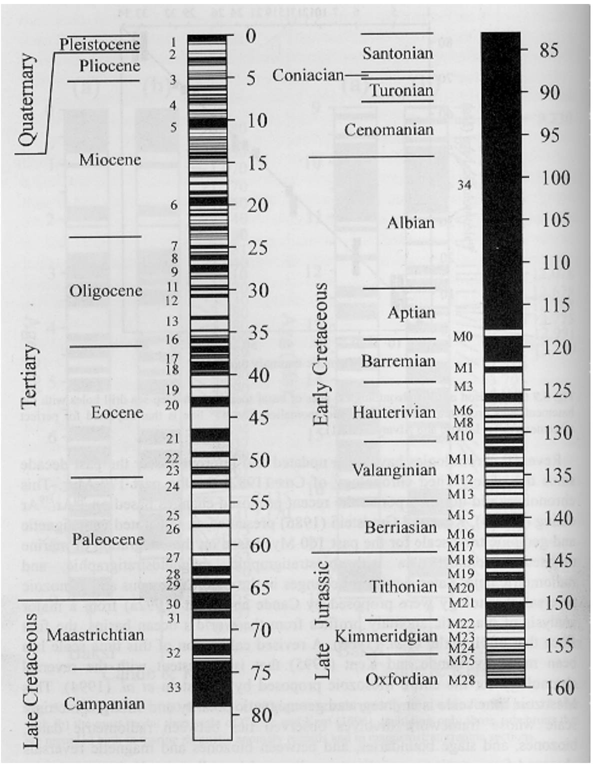
Solar winds



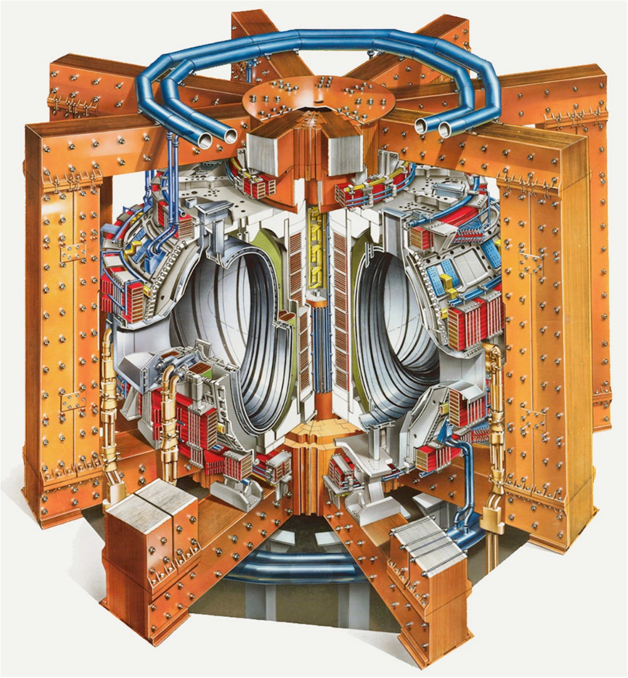
Stellar convection and dynamos



Periodic variation of sunspot

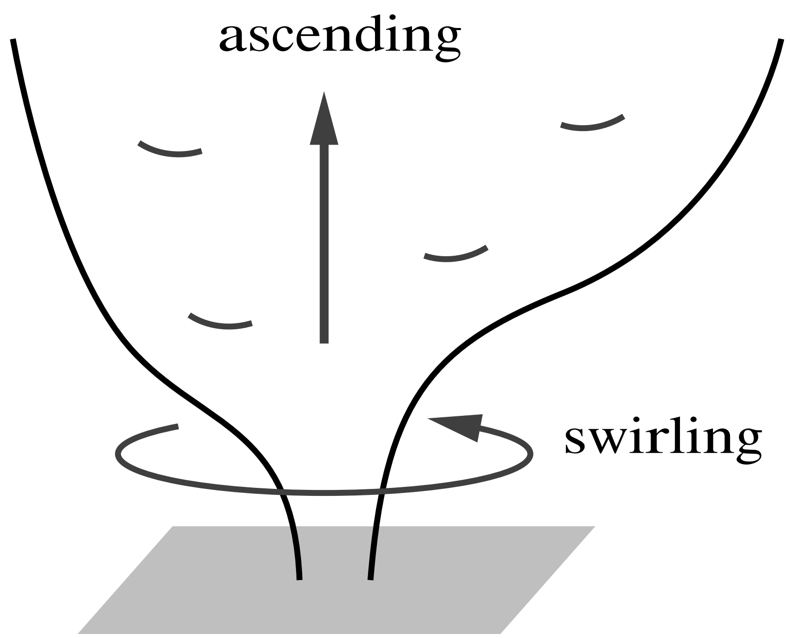


Polarity reversal of the geomagnetism

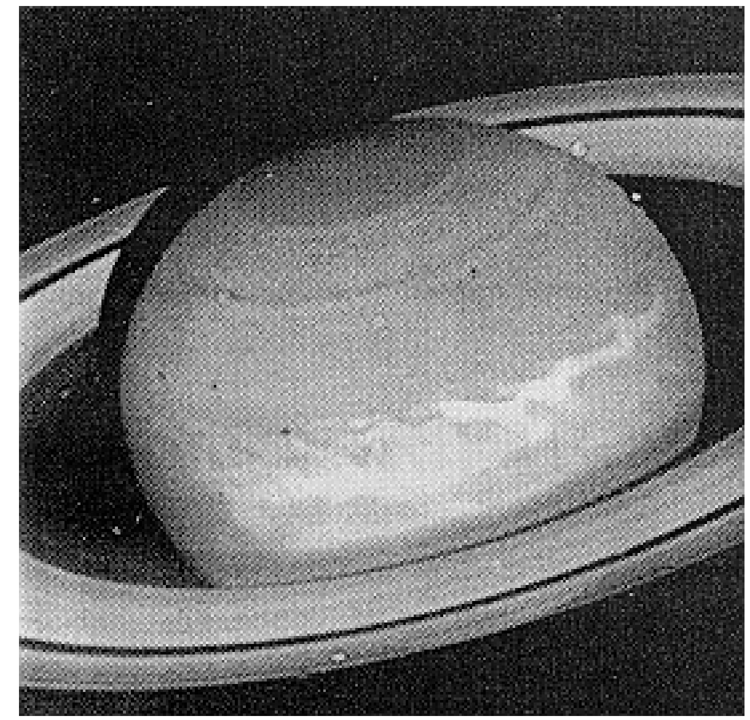


Joint European Torus (JET)

Vortex structure, large-scale magnetic field



Typhoon, tornado



White spot on Saturn

Structure formation ⟵ Breakage of symmetry

What suppresses turbulent viscosity and/or anomalous resistivity?

Suppression of cascade

Global magnetic fields of galaxies

Galactic magnetic field

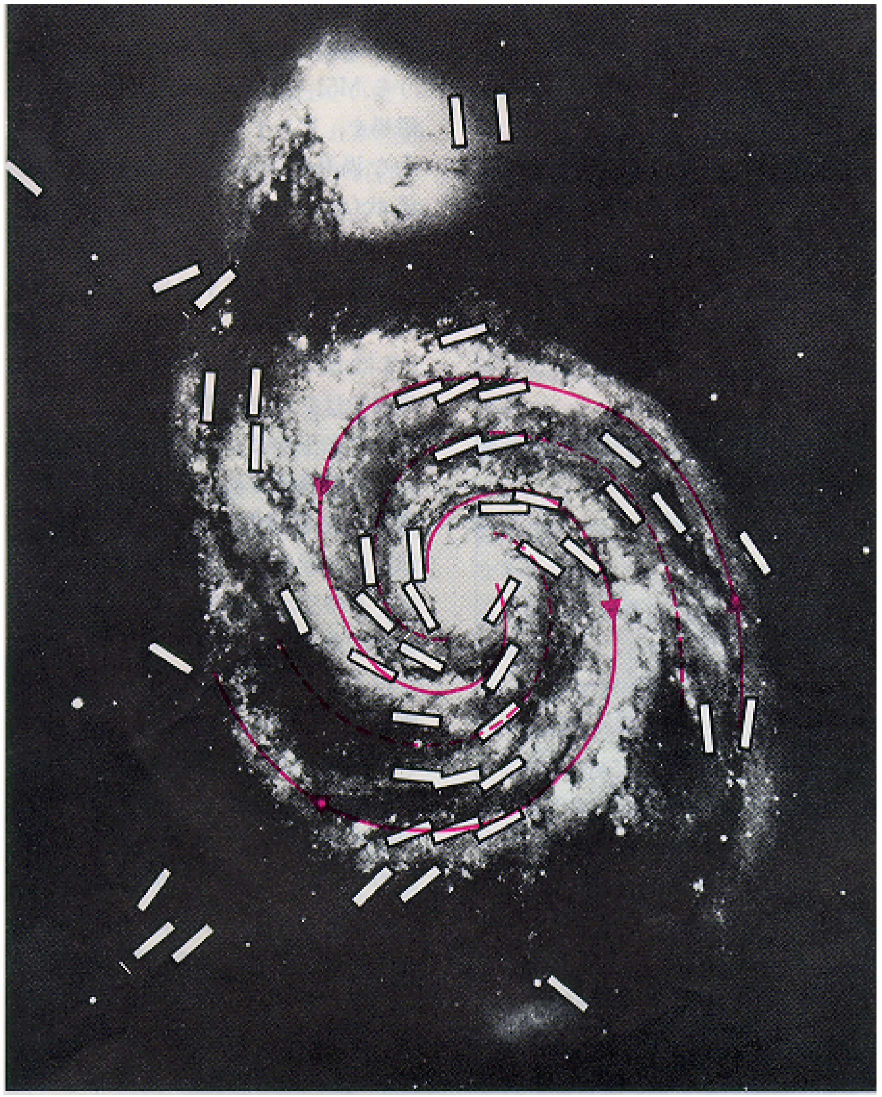
Toroidal magnetic field
is observable

Magnetic-field strength

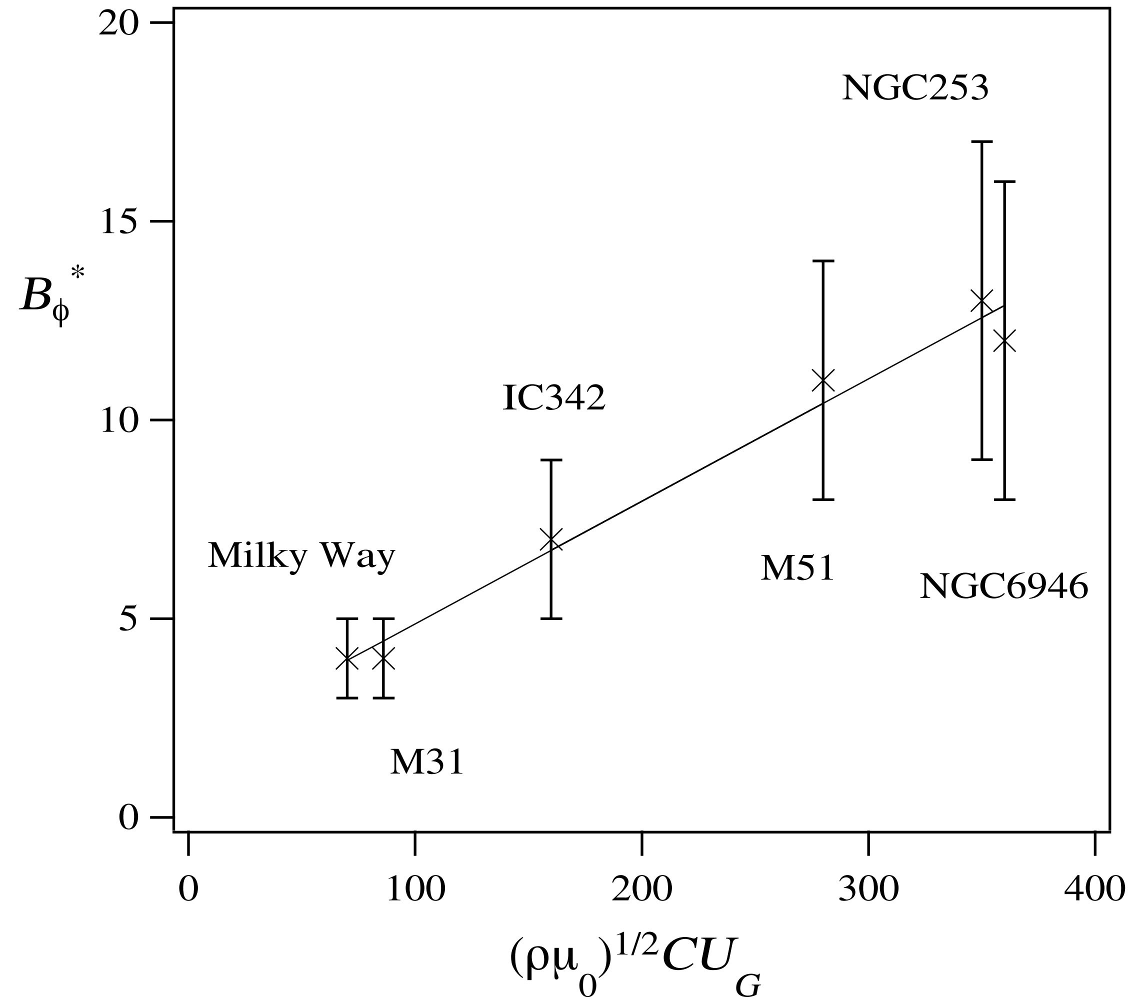
a few μG

Origin of galactic magnetic field

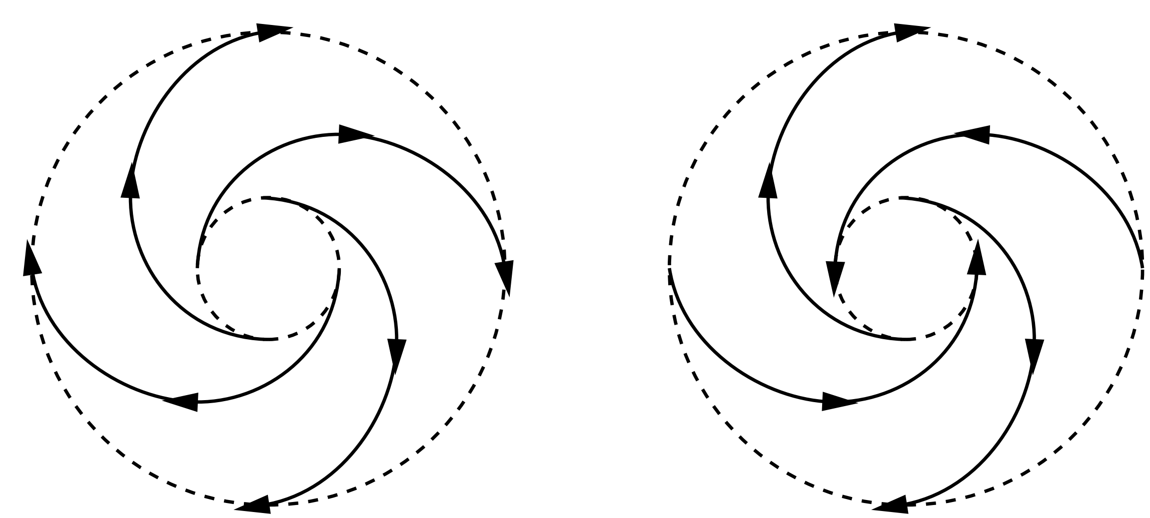
Primordial or Dynamo?



Magnetic field configuration in galaxy (M51)



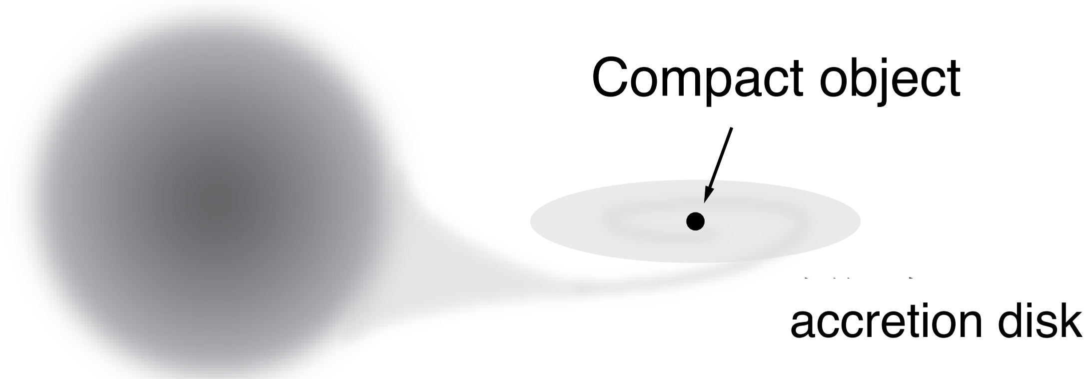
Rotational speed and magnetic-field strength of galaxies



ASS (axisymmetric spiral) fieldBSS (bisymmetric spiral) field

Accretion Disks

Plasmas
accreting to central compact object
with rotating



Astrophysical body	Compact object
Young stellar object	Protostar
Cataclysmic variables	White dwarf
X-ray binaries	Neutron star Black hole
Active galactic nuclei	Supermassive black hole

Conditions for accretion: Angular momentum loss

Turbulent viscosity Alfvén wave Jet

Astrophysical jets

Bipolar jets

Mechanism for driving jet

Rotation (Vortical motion)

↓

Toroidal magnetic-field generation
due to the cross-helicity effect

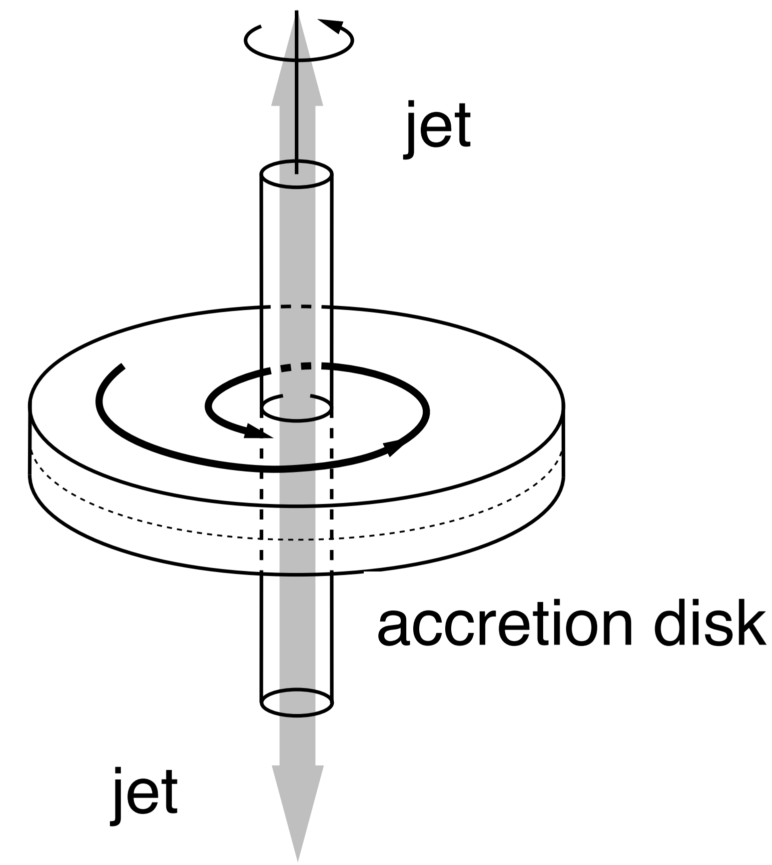
↓

Driving jet by magnetic energy

High collimation

Collimation O(1)/O(10^6)

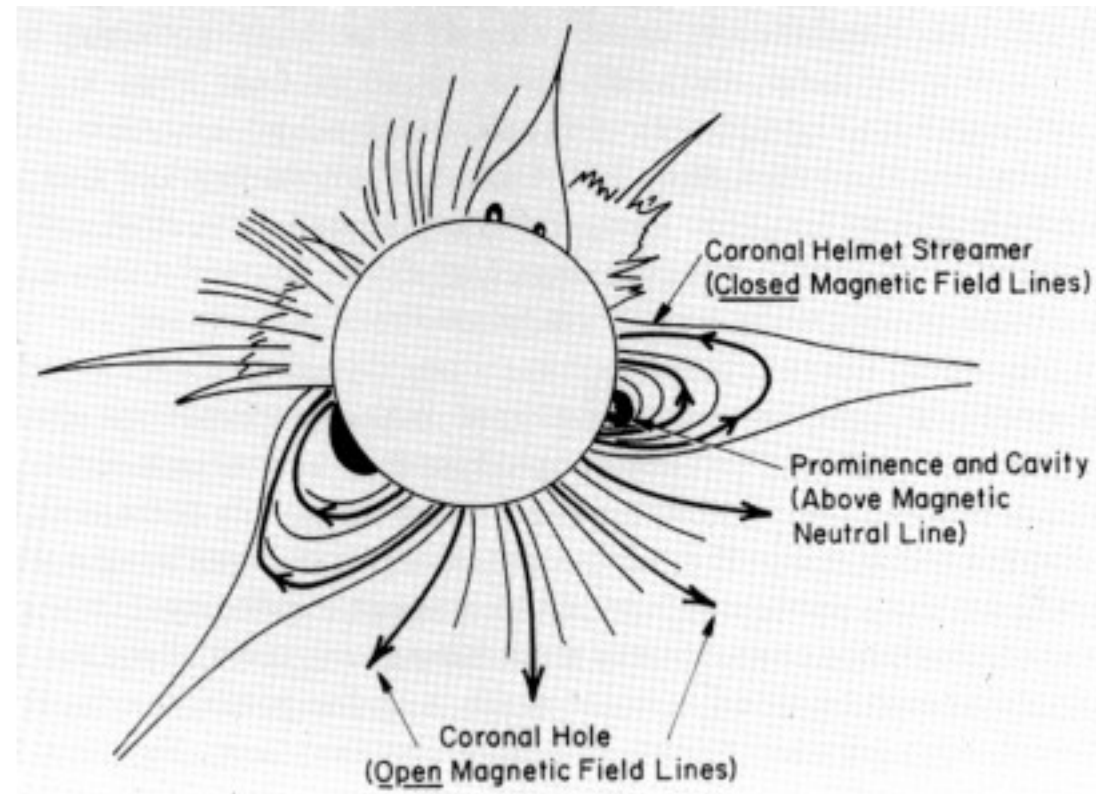
Magnetic confinement?



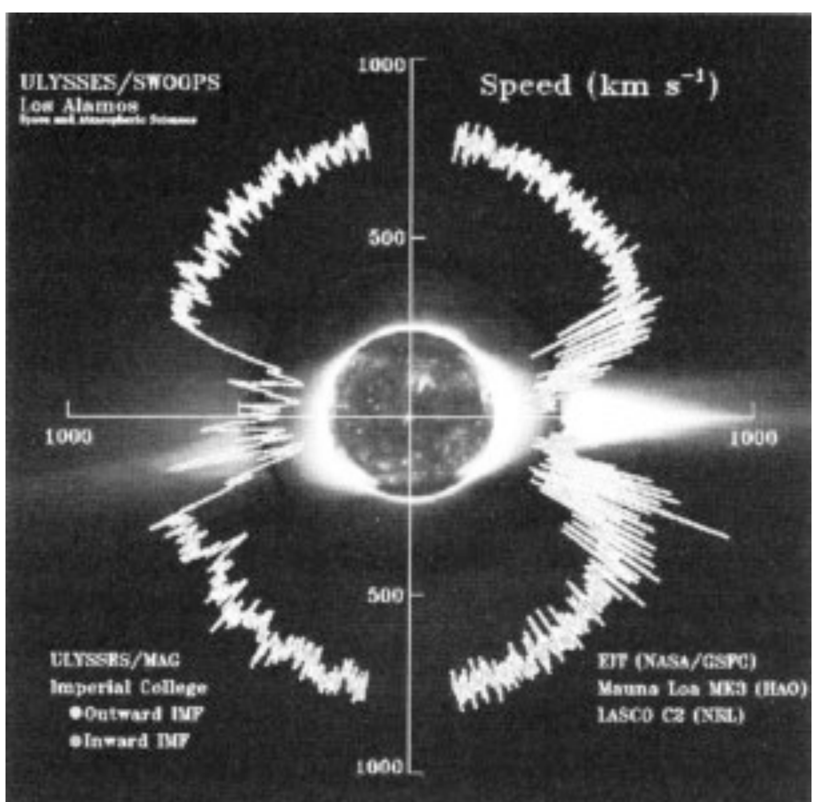
Jet from AGN 3C348

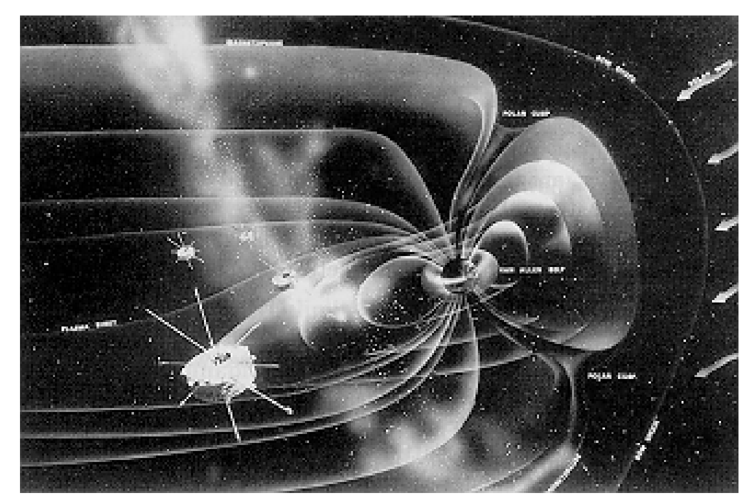
Solar Winds

High-speed plasma flow from stars

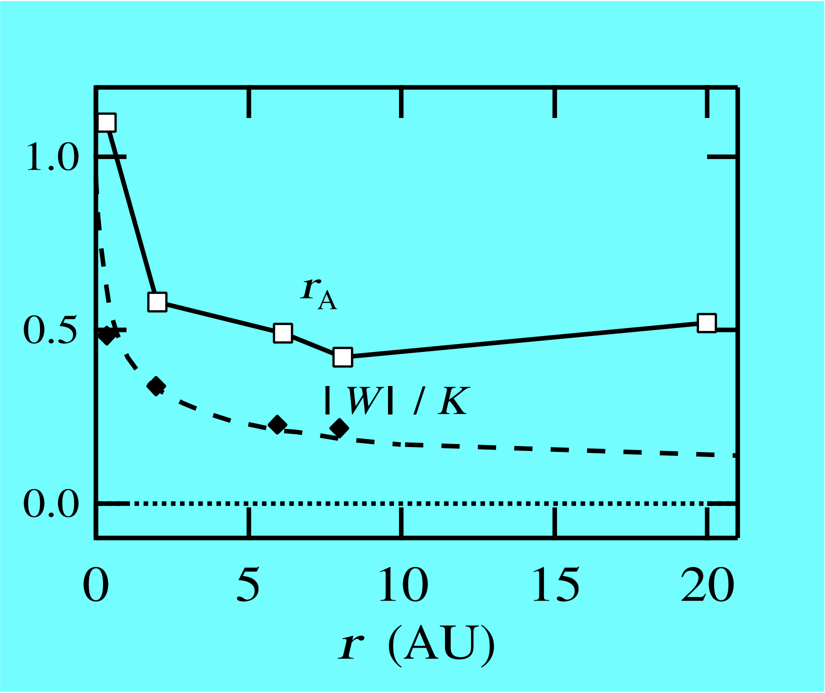


Inhomogeneous turbulence with cross helicity (velocity–magnetic-field correlation)





Evolution of solar-wind turbulence



cross correlation and Alfvén ratio against heliocentric distance (Roberts *et al.* 1990)

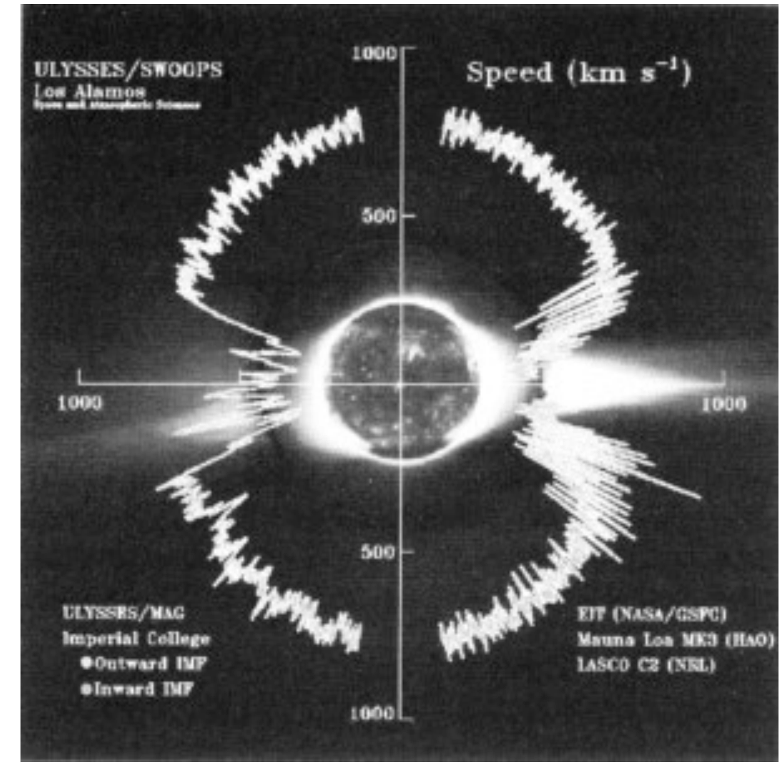
(i) Alfvén ratio

Alfvén ratio $r_{\rm A} \equiv \dfrac{\langle \mathbf{u}'^2 \rangle}{\langle \mathbf{b}'^2 \rangle} \simeq 0.5$ for $r \gtrsim 3\ \mathrm{AU}$

Near the Sun		Far from the Sun
Equipartition (Alfvénic)	⟶	Magnetic dominance (super Alfvénic)

(ii) Cross-helicity

Normalized cross helicity $\dfrac{|W|}{K} = \dfrac{|\langle \mathbf{u}' \cdot \mathbf{b}' \rangle|}{\langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle / 2}$



solar wind
(Ulysses mission, 1991-1996)

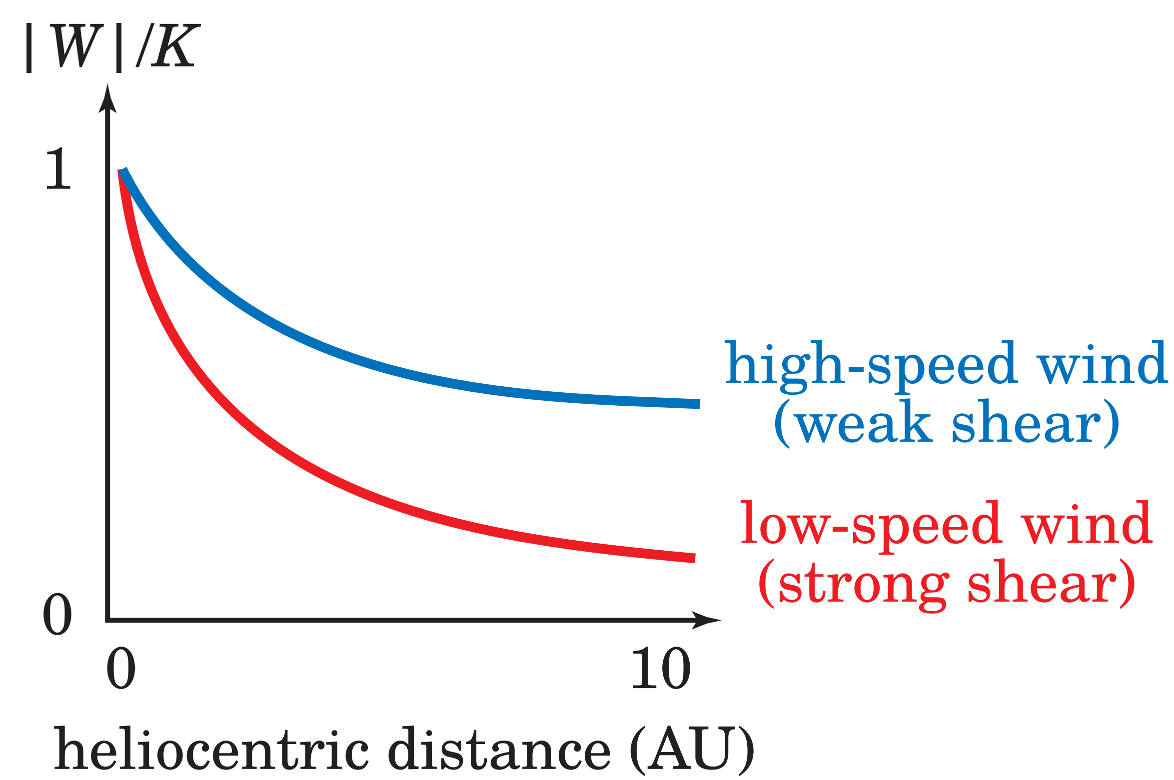
As the heliocentric distance increases

- High-speed wind
  from higher-latitude regions
  weak velocity shear

  cross helicity remains to be relatively large value of 0.4~0.7

- Low-speed wind
  from lower-latitude regions
  strong velocity shear

  cross helicity decreases to small value of 0.1~0.3

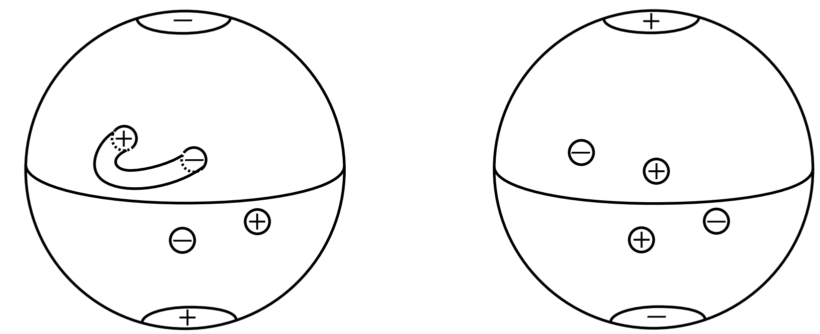


Magnetic activity of the Sun

Solar sunspot

Emergence of strong magnetic field (thousands G)

Toroidal magnetic field

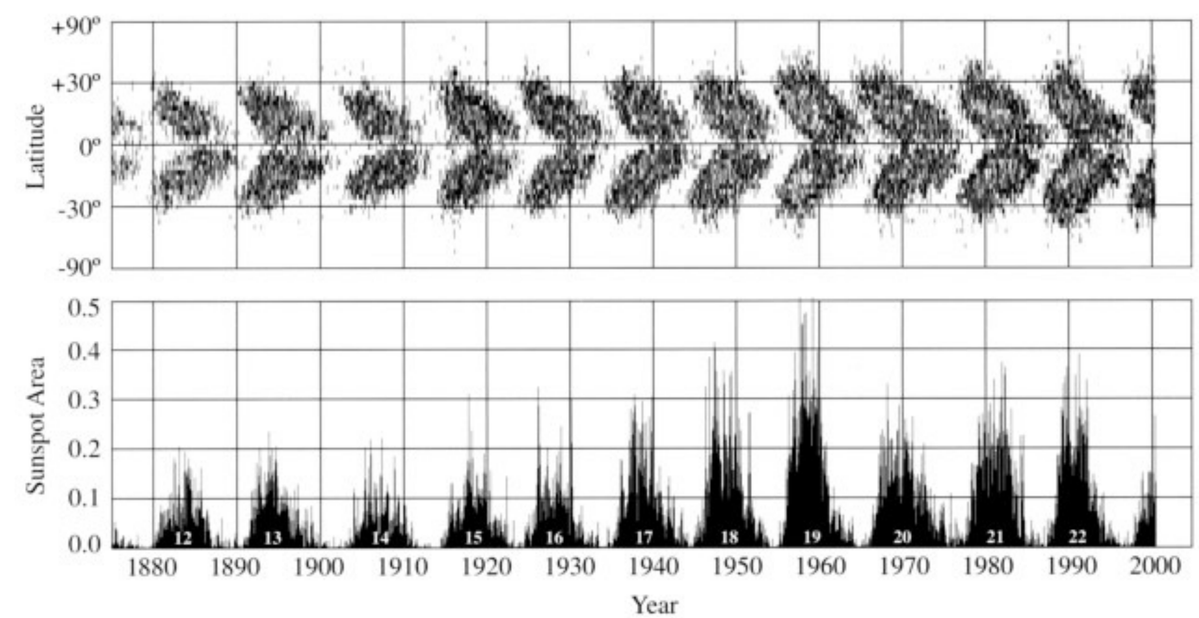


Polarity law of sunspot

Periodic behaviour of sunspot number

Periodic variation of magnetic field

Polarity reversal



Periodic variation of sunspot

What generates and sustains the solar magnetic field

⟵ Fluid plasma motions in the Sun

Solar interior

Core

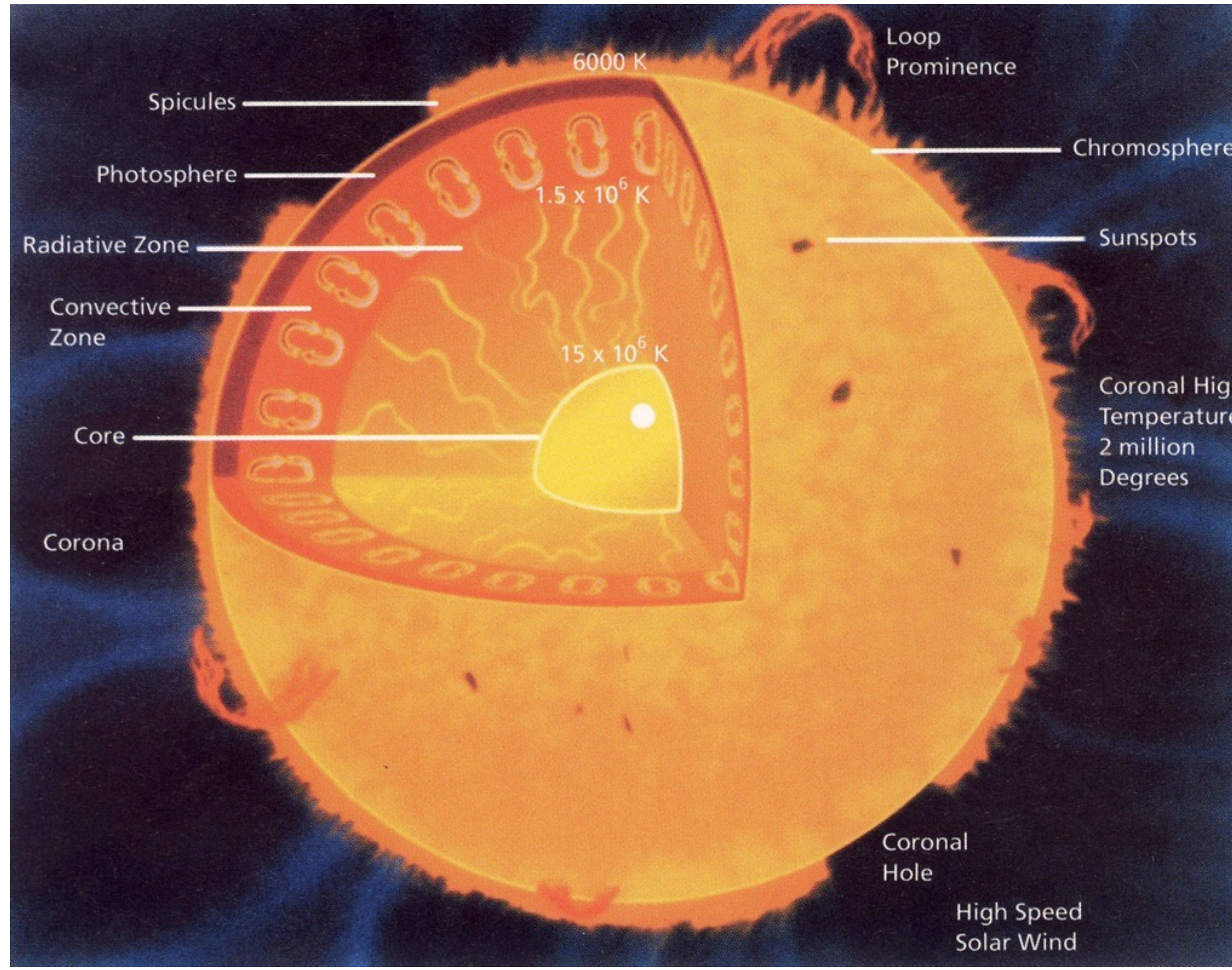
Thermal fusion

Radiative zone

Approximately rigid rotation

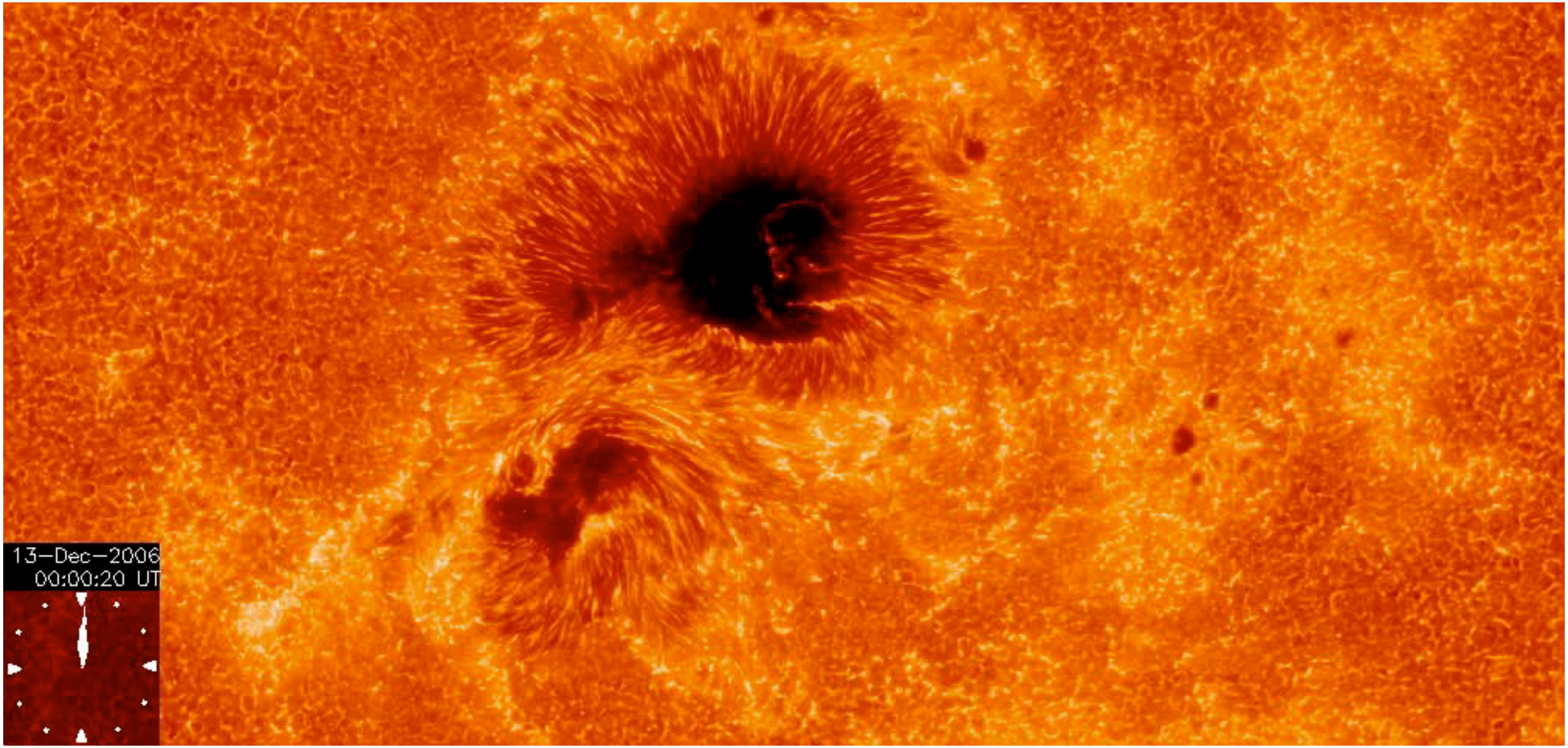
Convective zone

Differential rotation

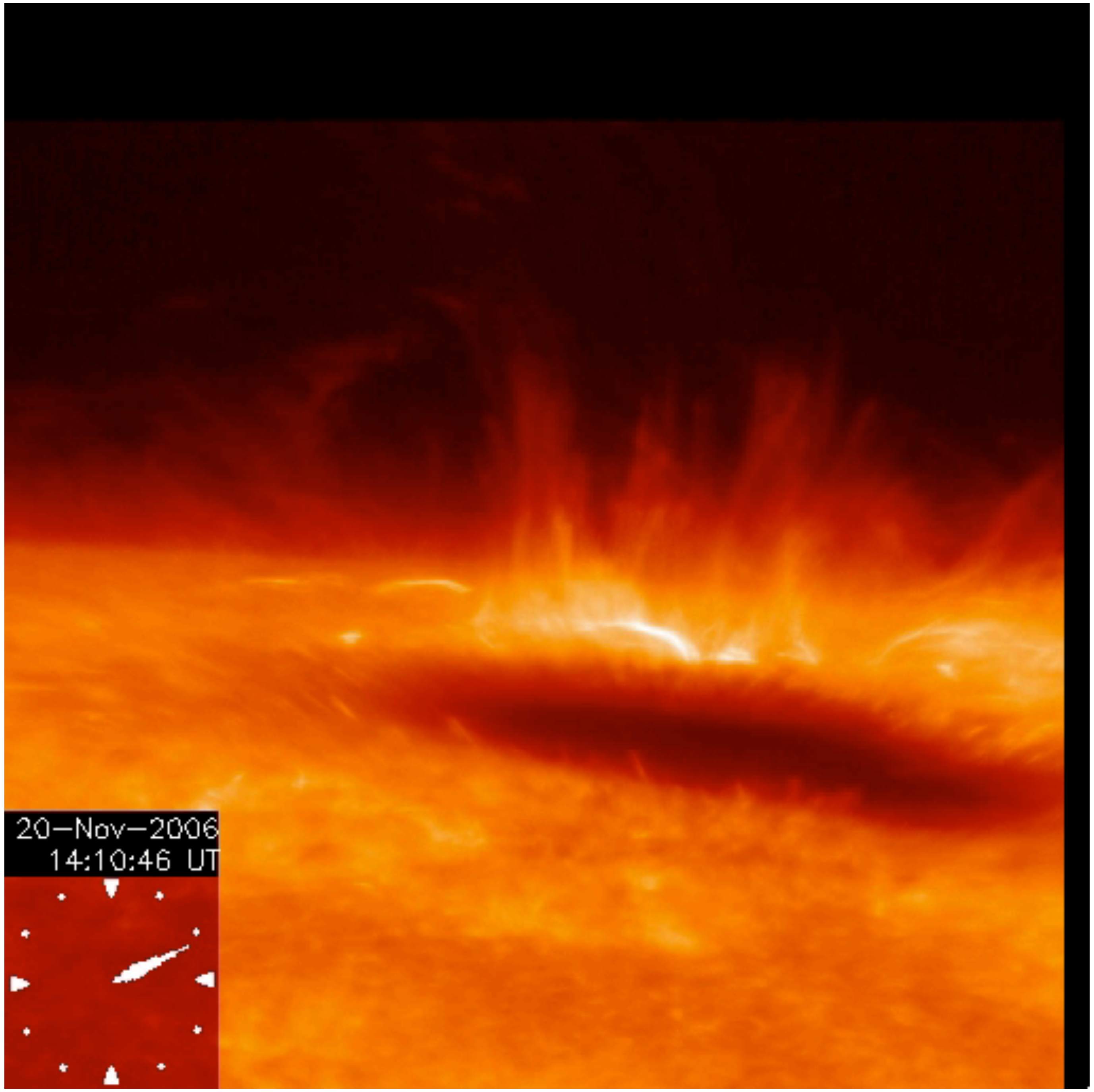


Motion in the convective zone → Turbulent electromotive force → Solar magnetic field

Okamoto,
NAOJ



Courtesy of the HINODE Group, NAOJ



Courtesy of the HINODE Group, NAOJ

Geomagnetism

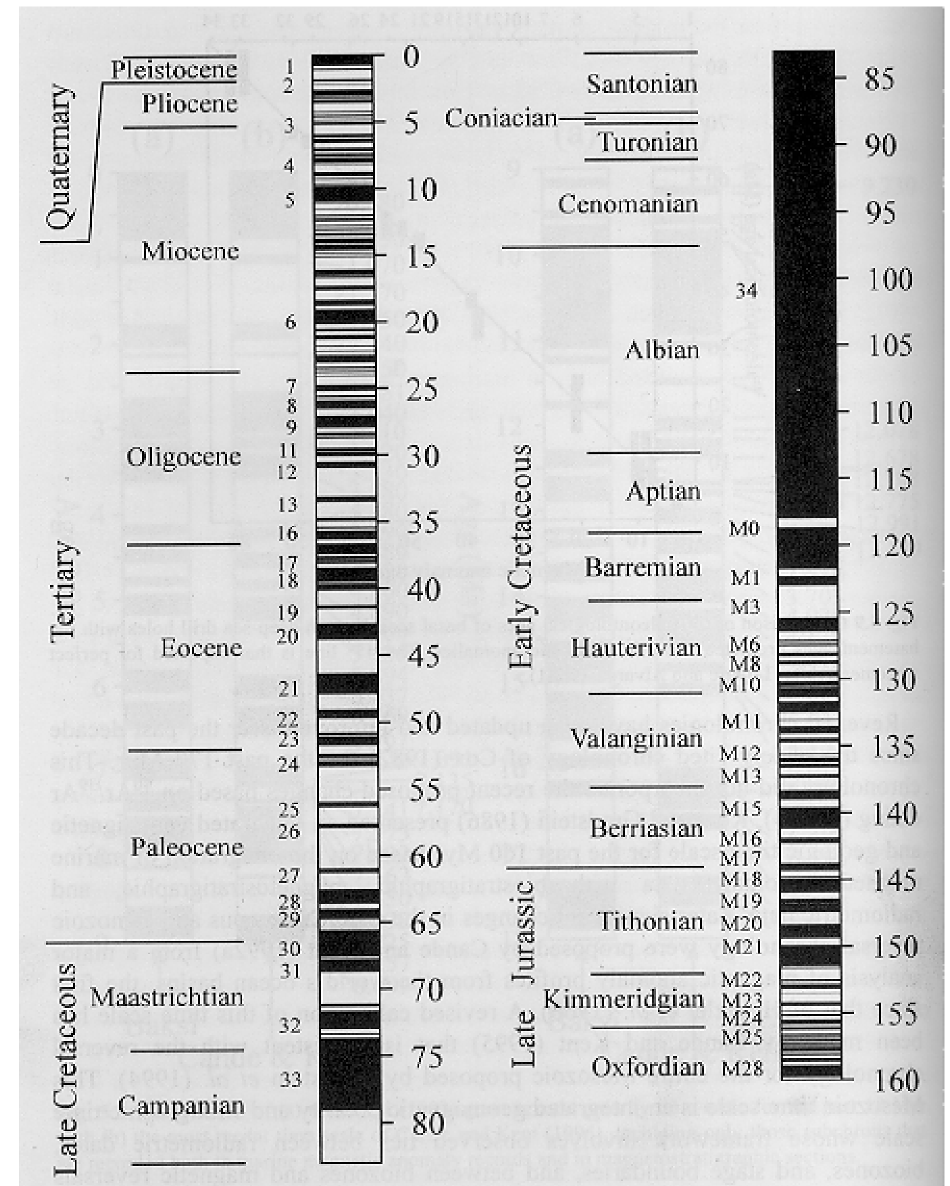
Polarity reversal of geomagnetic field

Irregular

Magnetic energy >> Kinetic energy

Motion of melted iron
in the outer core

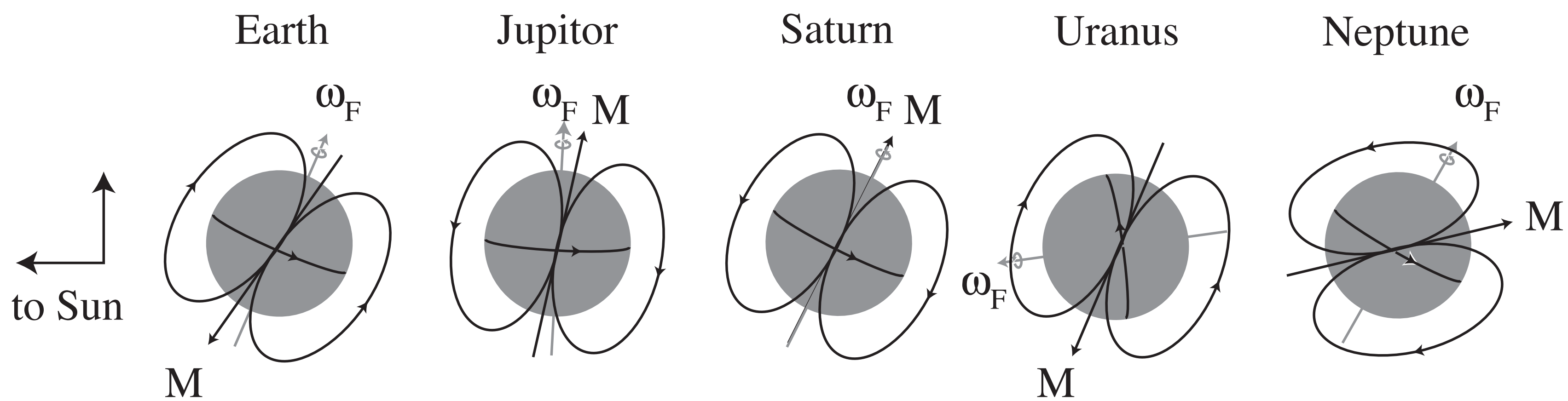
Only poloidal magnetic fields
are observable



Polarity reversal of the geomagnetism

Geomagnetism

Planetary magnetic field

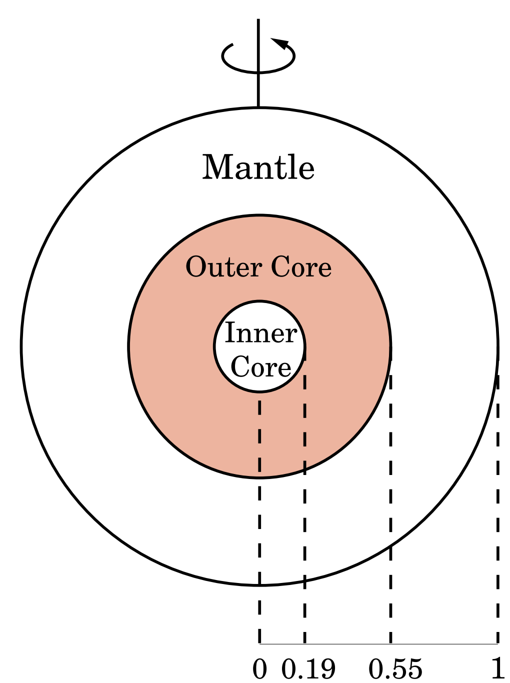


Directions of otation axis (ω_F) and dipole moment (M)

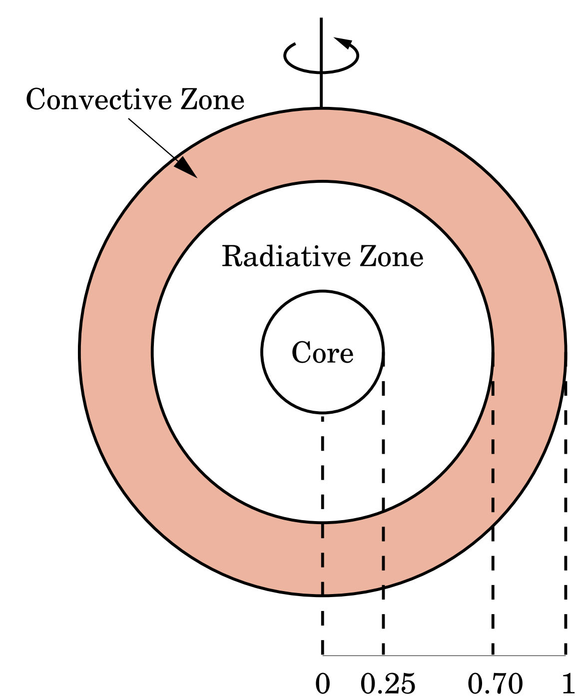
Differences

Earth Outer core
Melted iron Thick shell
Fast rotation
Irregular reversal

Sun Convection zone
Ionized H Thin shell
Relatively slow rotation
Periodic reversal



Earth's outer core



Solar convective zone

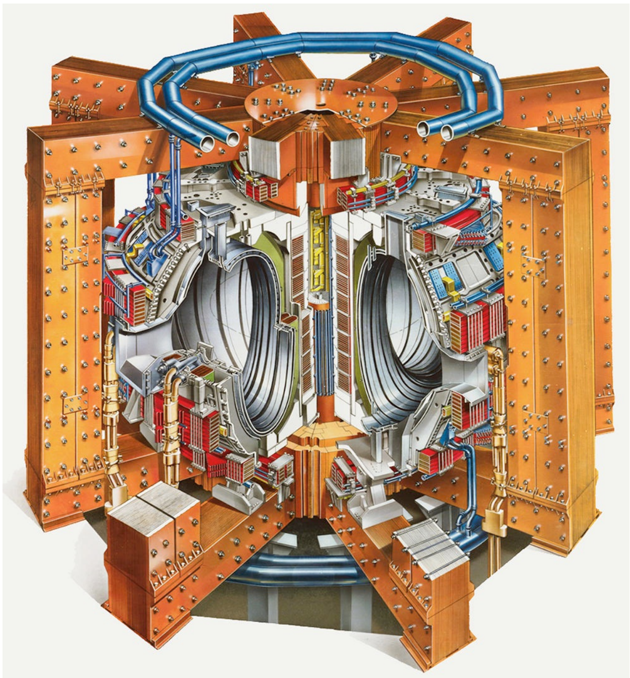
Fusion plasmas

Reversed field pinch

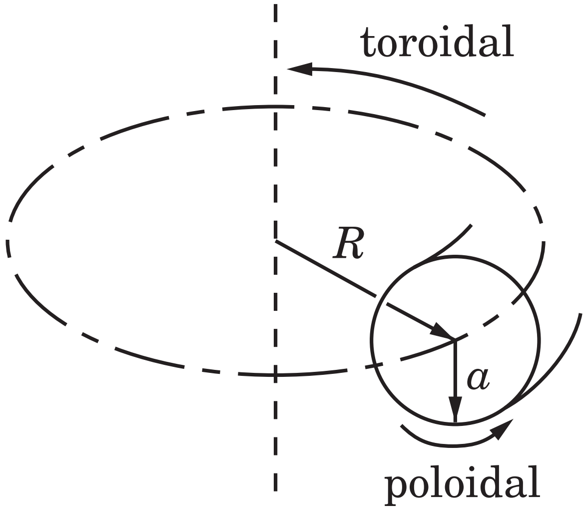
Tokamaks

Helical systems

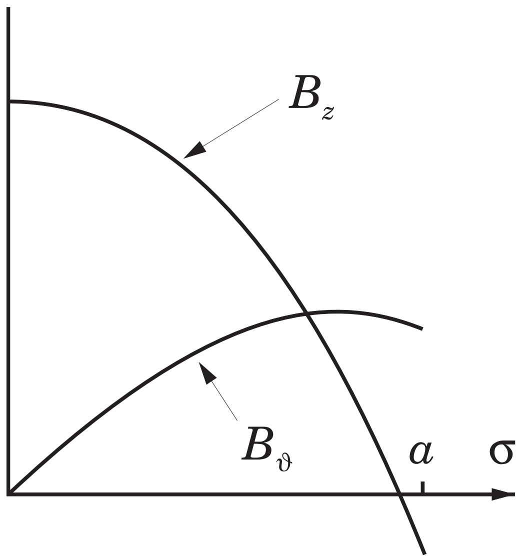
Inertial confinement



Joint European Torus (JET)



Poloidal and toroidal directions



Radial profiles of magnetic-field in RFP

Mean-field equations in compressible MHD

Yokoi, N., J. Plasma Phys. **84**, 735840501 & 775840603 (2018a,b)

Density

$$\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho}\mathbf{U}) = -\nabla \cdot \langle \rho' \mathbf{u}' \rangle$$

Turb. mass flux

Momentum

$$\frac{\partial}{\partial t}\overline{\rho}U^\alpha + \frac{\partial}{\partial x^a}\overline{\rho}U^a U^\alpha$$
$$= -(\gamma_0 - 1)\frac{\partial}{\partial x^\alpha}\overline{\rho}Q + \frac{\partial}{\partial x^\alpha}\mu S^{a\alpha} + (\mathbf{J}\times\mathbf{B})^\alpha$$
$$-\frac{\partial}{\partial x^\alpha}\left(\overline{\rho}\langle u'^a u'^\alpha \rangle - \frac{1}{\mu_0}\langle b'^a b'^\alpha \rangle + U^a \langle \rho' u'^\alpha \rangle + U^\alpha \langle \rho' u'^a \rangle\right) + R_U^\alpha$$

Reynolds stress

Turb. Maxwell stress

Turb. mass-energy correl.

Turb. energy flux

Internal energy

$$\frac{\partial}{\partial t}\overline{\rho}Q + \nabla \cdot (\overline{\rho}\mathbf{U}Q) = \nabla \cdot \left(\frac{\kappa}{C_V}\nabla Q\right) - \nabla \cdot (\overline{\rho}\langle q' \mathbf{u}' \rangle + Q\langle \rho' \mathbf{u}' \rangle + \mathbf{U}\langle \rho' q' \rangle)$$
$$-(\gamma_0 - 1)\left(\overline{\rho}Q\nabla\cdot\mathbf{U} + \overline{\rho}\langle q' \nabla \cdot \mathbf{u}' \rangle + Q\langle \rho' \nabla \cdot \mathbf{u}' \rangle\right) + R_Q$$

Turb. energy dilatation

Turb. mass dilatation

Magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U}\times\mathbf{B} + \langle \mathbf{u}' \times \mathbf{b}' \rangle) + \eta\nabla^2\mathbf{B}$$

Turb. electromotive force

Contents

- Turbulence properties
- Theoretical formulation
  - Non-linearity: Closure with response functions
  - Inhomogeneities, anisotropies, non-equilibrium properties: Multiple-scale analysis
- Illustrative examples
  - Angular-momentum transport by inhomogeneous kinetic helicity
  - Convective transport
    - Plumes as coherent fluctuations
    - Non-equilibrium properties along coherent motions
  - Dynamos
- Turbulence modelling

Turbulence properties

群盲撫象　葛飾北斎 Hokusai

Turbulence

Nonlinear

Cascade

Dissipative

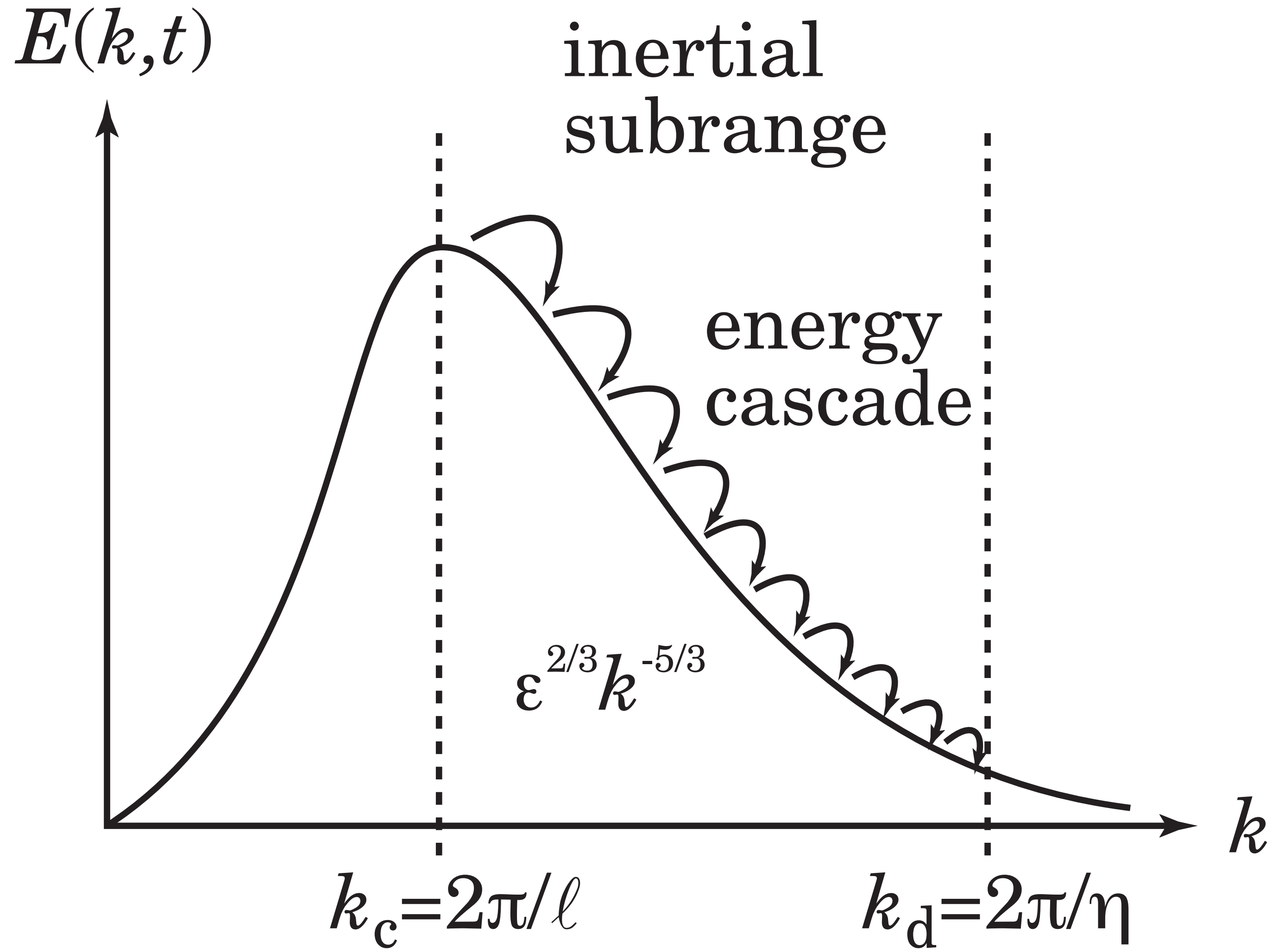
Anomalous transport

Local vs. non-local

Homogeneous
vs. inhomogeneous

Isotropic vs. anisotropic

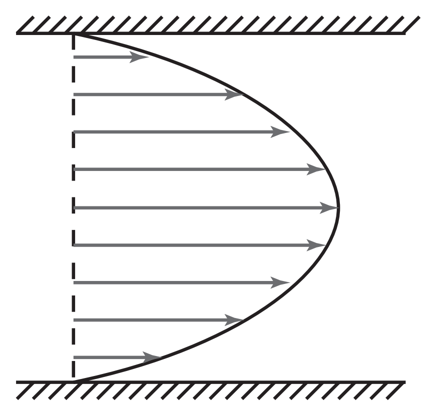
Equilibrium vs. non-equilibrium



Simple picture of cascade

Effects of turbulence

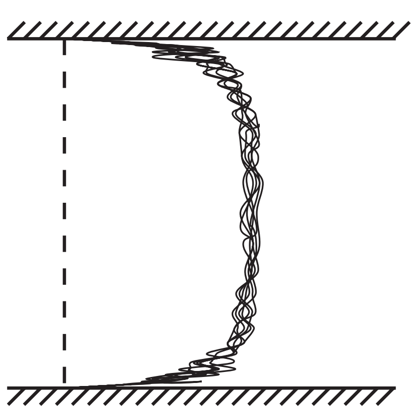
Laminar pipe flow



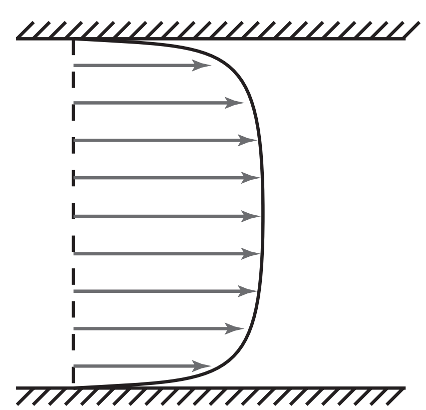
$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}$$

Turbulent pipe flow

Large-scale structure destroyed



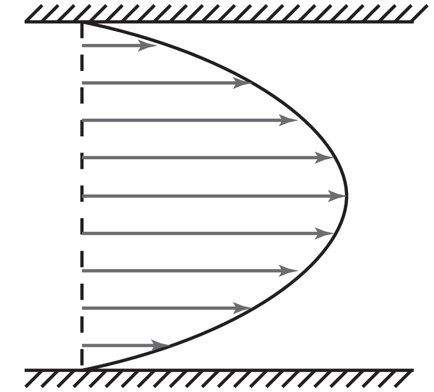
Instantaneous



Mean

$$\begin{cases} \mathbf{u} = \mathbf{U} + \mathbf{u}', & \mathbf{U} = \langle \mathbf{u} \rangle \\ p = P + p', & P = \langle p \rangle \end{cases}$$

Enhancement of transport



Laminar

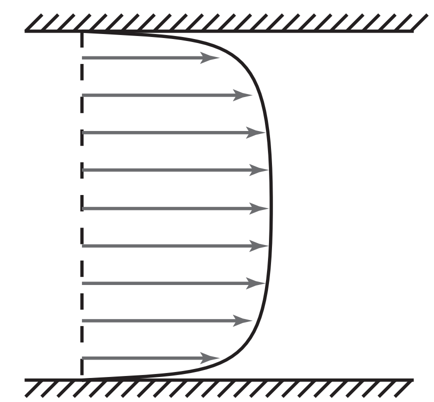
Equation of mean velocity

$$\frac{DU_\alpha}{Dt} \equiv \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_a}\right) U_\alpha = -\frac{\partial P}{\partial x_\alpha} - \frac{\partial}{\partial x_a}\langle u'_a u'_\alpha \rangle + \nu \frac{\partial^2 U_\alpha}{\partial x_a^2}$$

Reynolds stress $$\langle u'_\alpha u'_\beta \rangle = \frac{2}{3} K \delta_{\alpha\beta} - \nu_{\mathrm{T}} \left(\frac{\partial U_\alpha}{\partial x_\beta} + \frac{\partial U_\beta}{\partial x_\alpha}\right)$$ (Model)

ν_{T} : eddy viscosity (turbulent viscosity) (Boussinesq, 1877)

$$\longrightarrow \quad \frac{\partial U_\alpha}{\partial t} + U_a \frac{\partial U_\alpha}{\partial x_a} = -\frac{\partial P}{\partial x_\alpha} + \frac{\partial}{\partial x_a}\left[(\nu + \nu_{\mathrm{T}})\left(\frac{\partial U_\alpha}{\partial x_a} + \frac{\partial U_a}{\partial x_\alpha}\right)\right]$$



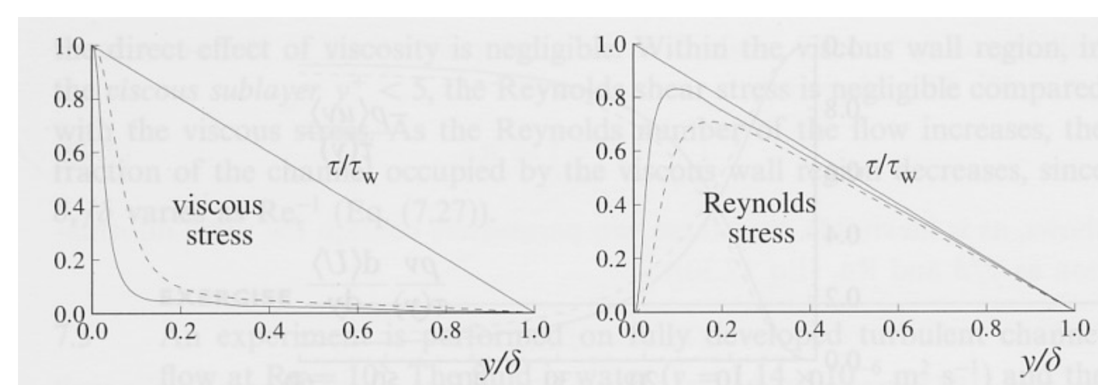
Turbulent

Enhancement of transport

$$\frac{\text{(Turbulent viscosity)}}{\text{(Molecular viscosity)}} \quad \frac{\nu_{\mathrm{T}}}{\nu} \sim \frac{u\ell}{\nu} \sim Re$$

Variation in space and time

DNS of turbulent and viscous stresses

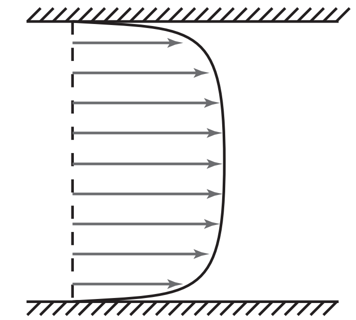


dashed line, Re=5,600; solid line Re=13,750

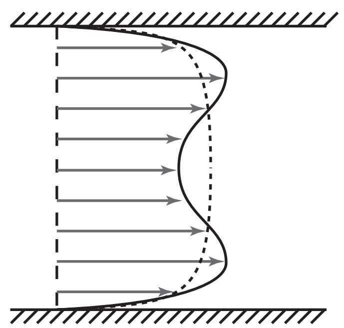
Suppression of transport

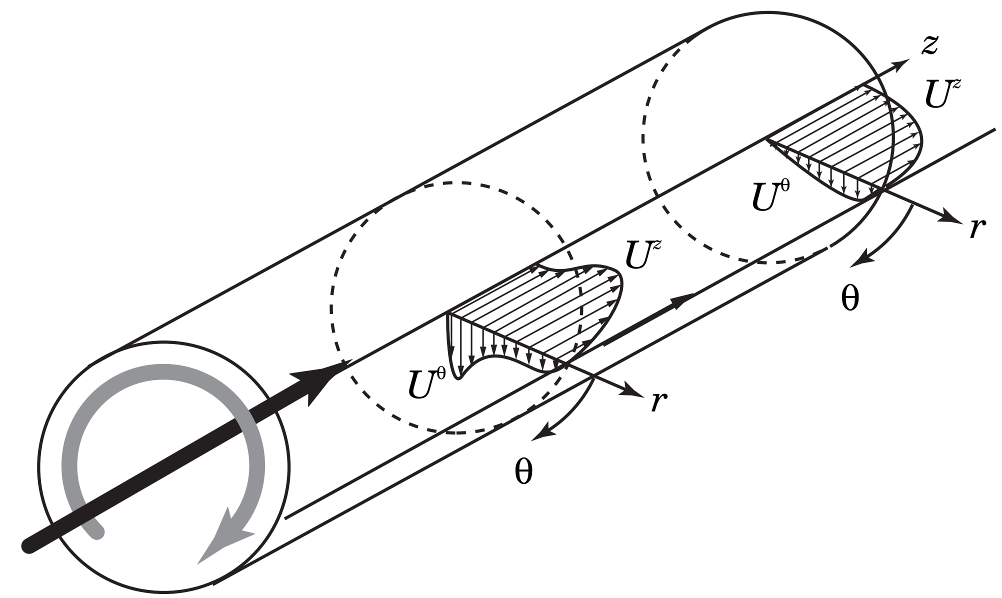
Swirling flow in a circular pipe

Turbulent

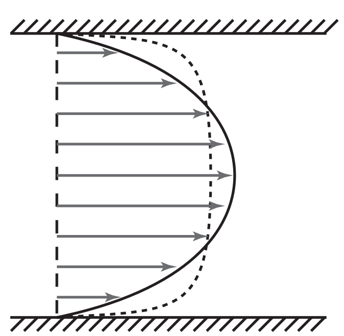


Turbulent swirling pipe flow





Axially rotating turbulent pipe flow



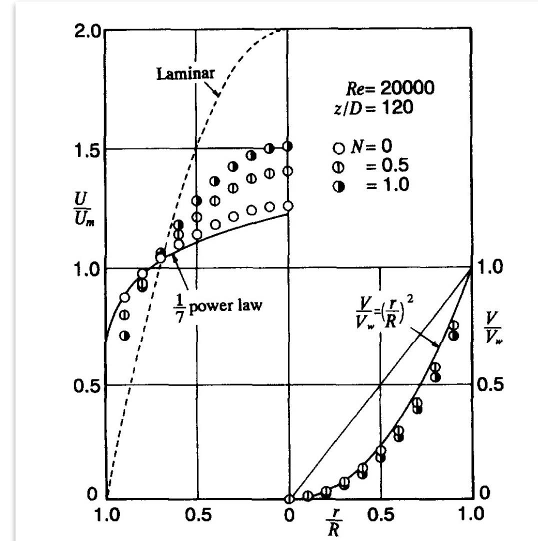


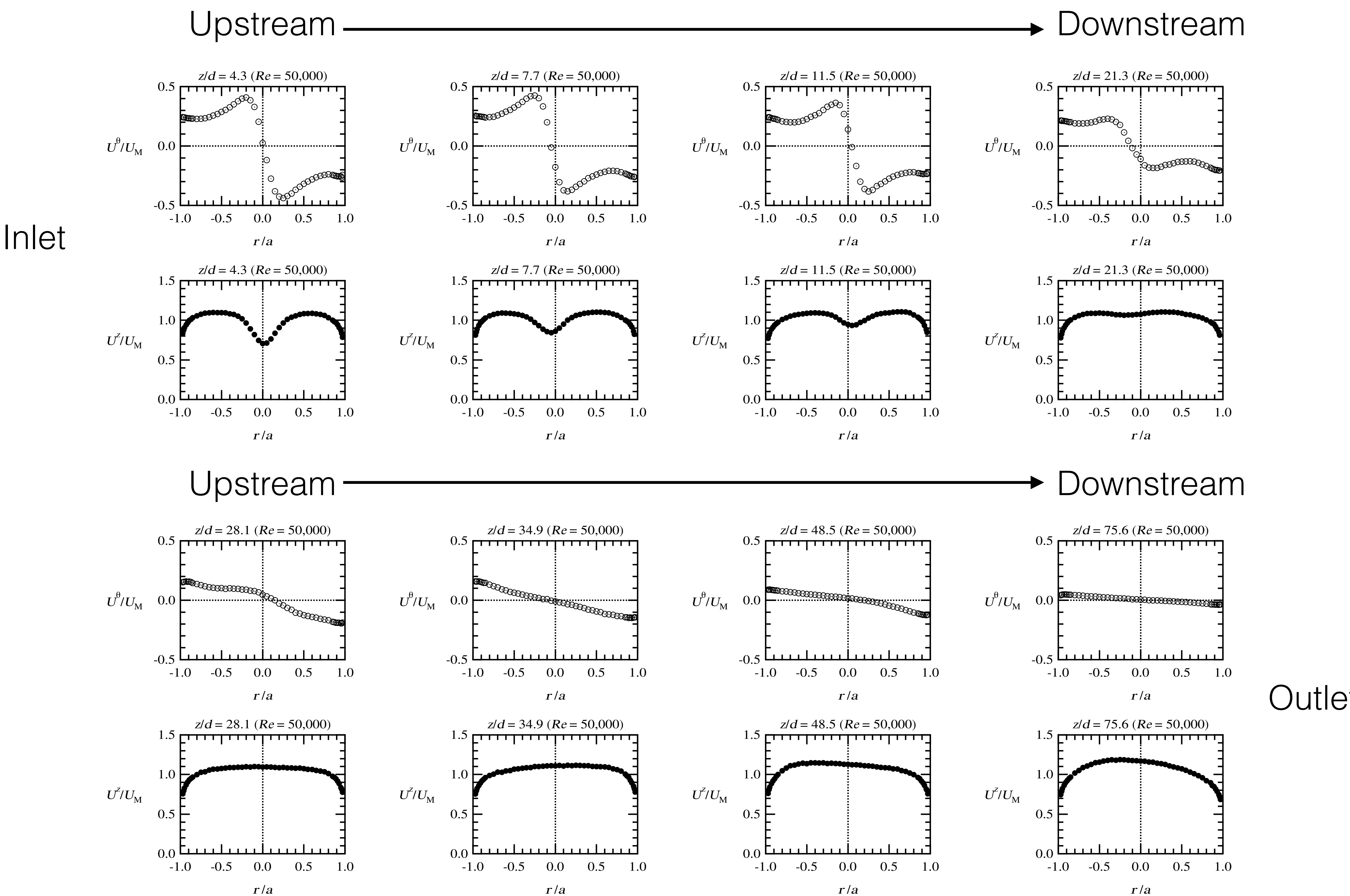
Figure 4 Time-mean velocity profiles (uncertainties in r/R are ±1%, and in U/U_m and V/V_w are ±2%)

(Imao *et al*. 1996)

These flow properties cannot be reproduced
by the standard eddy-viscosity representation at all.
Too much dissipative.

Axial evolution of a weak swirl

(Steenbergen, 1995)



Equation of fluctuating velocity

$$\mathbf{u} = \mathbf{U} + \mathbf{u}', \quad \mathbf{U} = \langle \mathbf{u} \rangle, \quad \mathbf{u}' = \mathbf{u} - \langle \mathbf{u} \rangle$$

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial U_\alpha}{\partial x_a} - u'_a \frac{\partial u'_\alpha}{\partial x_a} + \frac{\partial}{\partial x_a} \langle u'_a u'_\alpha \rangle - \frac{\partial p'}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

turbulence–mean velocity interaction

turbulence–turbulence interaction

→ Instability approach

Quasi-linear -> nonlinearity

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial U_\alpha}{\partial x_a} - \frac{\partial p'^{(\mathrm{R})}}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

Linear in $\mathbf{u}'$ and $p'^{(\mathrm{R})}$, each (Fourier) mode evolves independently

→ Closure approach

Homogeneous isotropic -> inhomogeneities

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial u'_\alpha}{\partial x_a} + \frac{\partial}{\partial x_a} \langle u'_a u'_\alpha \rangle - \frac{\partial p'^{(\mathrm{S})}}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

Homogeneous turbulence, no dependence on large-scale inhomogeneity

Navier–Stokes equation in the wave-number space

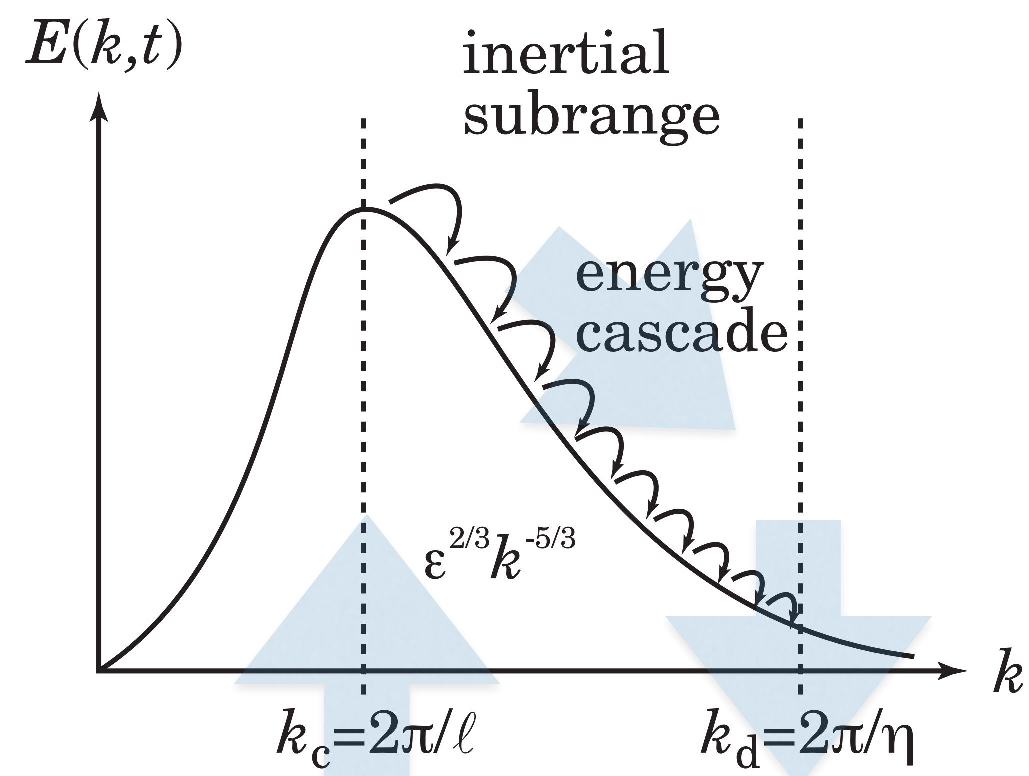
Homogeneous and Isotropic Turbulence (HIT)

$$\frac{\partial \hat{u}_\alpha(\mathbf{k};t)}{\partial t} - ik_a \iint d\mathbf{p} d\mathbf{q} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \hat{u}_a(\mathbf{p};t) \hat{u}_\alpha(\mathbf{q};t) = ik_\alpha \hat{p}(\mathbf{k};t) - \nu k^2 \hat{u}_\alpha(\mathbf{k};t)$$

The dynamics of **k** mode is governed by its interaction with all other modes

↓

All the scales, from the largest to smallest scales, have to be solved



Largest scale (integral scale)

$\ell = \ell\{K, \varepsilon\}$

Smallest scale (viscous scale)

$\eta = \eta\{\nu, \varepsilon\}$

Energy injection

Energy flux

Energy dissipation

$\mathcal{E}$

Kolmogorov microscale

$$\eta = \eta\{\nu, \varepsilon\}$$

$$u_\eta \sim \frac{\nu}{\eta} \qquad \left[Re = \frac{u_\eta \eta}{\nu} \sim 1\right]$$

$$\eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \quad \longleftarrow \quad \varepsilon \sim \nu \frac{u_\eta^2}{\eta^2} \sim \nu \frac{(\nu/\eta)^2}{\eta^2} \sim \frac{\nu^3}{\eta^4}$$

$$u_\eta = (\varepsilon \nu)^{1/4} \qquad u_\eta \sim \frac{\nu}{\eta} \sim \frac{\nu}{(\nu^3/\varepsilon)^{1/4}} \sim (\varepsilon\nu)^{1/4}$$

$$\tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{1/2} \qquad \tau_\eta \sim \frac{\eta}{u_\eta} \sim \frac{(\nu^3/\varepsilon)^{1/4}}{(\varepsilon\nu)^{1/4}} \sim \left(\frac{\nu}{\varepsilon}\right)^{1/2}$$

Integral scale

$$\ell = \ell\{K, \varepsilon\}$$

$$\ell \sim \frac{u_\ell^3}{\varepsilon} \sim \frac{K^{3/2}}{\varepsilon} \quad \longleftarrow \quad \varepsilon \sim \frac{u_\ell^2}{\ell/u_\ell} \sim \frac{u_\ell^3}{\ell} \qquad Re = \frac{u_\ell \ell}{\nu}$$

$$u_\ell \sim K^{1/2}$$

$$\tau_\ell \sim \frac{K}{\varepsilon} \qquad \varepsilon \sim \frac{K}{\tau_\ell}$$

Scale differences

Length

$$\frac{\eta}{\ell} \sim Re^{-3/4}$$

$$\frac{\eta}{\ell} \sim \frac{(\nu^3/\varepsilon)^{1/4}}{K^{3/2}/\varepsilon} \sim \frac{(\varepsilon\nu)^{3/4}}{K^{3/2}} \sim \left(\frac{\varepsilon\nu}{K^2}\right)^{3/4} \sim \left(\frac{\nu}{u_\ell \ell}\right)^{3/4} \sim Re^{-3/4}$$

Velocity

$$\frac{u_\eta}{u_\ell} \sim Re^{-1/4}$$

$$\frac{u_\eta}{u_\ell} \sim \frac{(\varepsilon\nu)^{1/4}}{K^{1/2}} \sim \left(\frac{\varepsilon\nu}{K^2}\right)^{1/4} \sim \left(\frac{\nu}{u_\ell \ell}\right)^{1/4} \sim Re^{-1/4}$$

Time

$$\frac{\tau_\eta}{\tau_\ell} \sim Re^{-1/2}$$

$$\frac{\tau_\eta}{\tau_\ell} \sim \frac{(\nu/\varepsilon)^{1/2}}{K/\varepsilon} \sim \left(\frac{\varepsilon\nu}{K^2}\right)^{1/2} \sim \left(\frac{\nu}{u_\ell \ell}\right)^{1/2} \sim Re^{-1/2}$$

$$\left[\nu_{\mathrm{T}} \sim u_\ell \ell, \qquad \frac{\nu_{\mathrm{T}}}{\nu} \sim \frac{u_\ell \ell}{\nu} \sim Re\right]$$

Largest and smallest scales in turbulence

Largest scale (integral scale) $\ell = \ell\{K, \varepsilon\}$ $\ell \sim \frac{u^3}{\varepsilon} \sim \frac{K^{3/2}}{\varepsilon}$

Smallest scale (viscous scale) $\eta = \eta\{\nu, \varepsilon\}$ $\eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$

$\frac{\text{(Largest scale)}}{\text{(Smallest scale)}}$

$$\frac{\ell}{\eta} \sim \left(\frac{u\ell}{\nu}\right)^{3/4} = O\left(Re^{3/4}\right)$$

Required grid points

$$N_{\mathrm{G}} = \left(\frac{\ell}{\eta}\right)^3 = O\left(Re^{9/4}\right)$$

	Re	N_G
Walking	$O(10^4)$	$O(10^9)$
Cars	$O(10^6)$	$O(10^{13.5})$
Airplanes	$O(10^8)$	$O(10^{18})$

	Re	N_G
Earth's outer core	$O(10^8)$	$O(10^{18})$
Solar convection zone	$O(10^{10})$	$O(10^{22.5})$
Galaxies	$O(10^{11})$	$O(10^{25})$

Direct numerical simulation (DNS) is just impossible in the foreseeable future

Theoretical formulation

Approaches to turbulence

Equation of fluctuating velocity

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial U_\alpha}{\partial x_a} - u'_a \frac{\partial u'_\alpha}{\partial x_a} + \frac{\partial}{\partial x_a} \langle u'_a u'_\alpha \rangle - \frac{\partial p'}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

turbulence–mean interaction | turbulence–turbulence interaction

Instability approach → Quasi-linear theory (+ non-linearity)

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial U_\alpha}{\partial x_a} - \frac{\partial p^{(\mathrm{R})}}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

Linear in $\mathbf{u}'$ and $p'^{(\mathrm{R})}$, each (Fourier) mode evolves independently

Closure approach → Homogeneous turb. theory (+ multiple scale)

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial u'_\alpha}{\partial x_a} + \frac{\partial}{\partial x_a} \langle u'_a u'_\alpha \rangle - \frac{\partial p'^{(\mathrm{S})}}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

Homogeneous turbulence, no dependence on large-scale inhomogeneity

Nonlinearity

Renormalised perturbation expansion theory

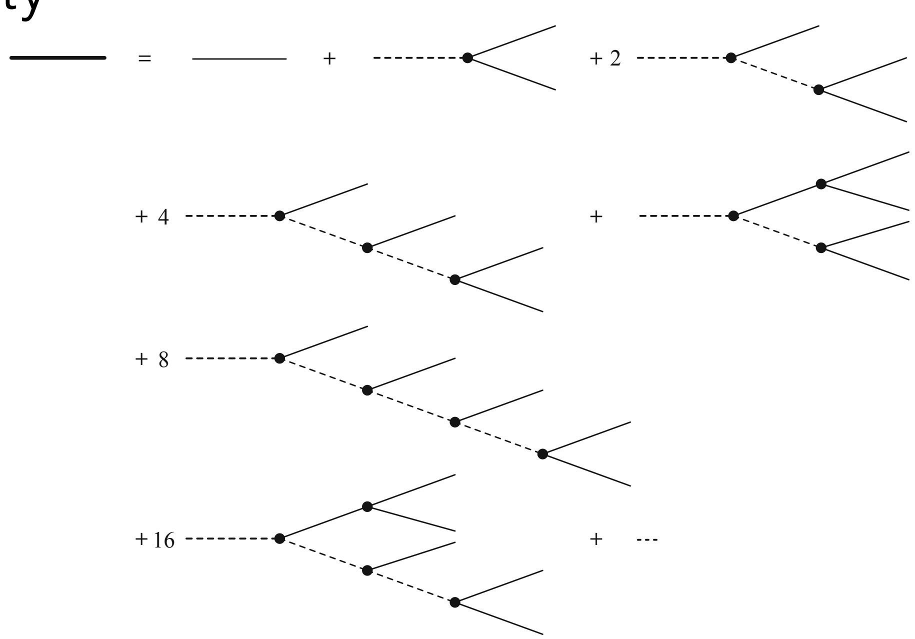
Kraichnan, R. H. (1959) "The structure of isotropic turbulence **at very high Reynolds number**," J. Fluid Mech. **5**, 497

Velocity
$$u_\alpha(\mathbf{k};t) = u_\alpha^{(\mathrm{L})}(\mathbf{k};t) + iM_{cab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$
$$\times \int_{-\infty}^{t} dt_1 G_{\alpha c}^{(\mathrm{L})}(\mathbf{k};t,t_1) u_a(\mathbf{p};t_1) u_b(\mathbf{q};t_1)$$

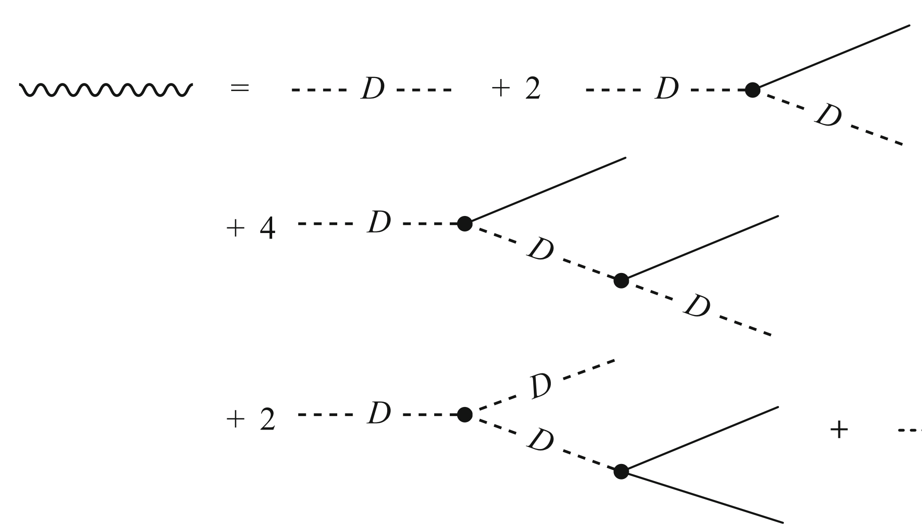
Response function
$$G'_{\alpha\beta}(\mathbf{k};t,t') = G_{\alpha\beta}^{(\mathrm{L})}(\mathbf{k};t,t') + i2M_{cab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$
$$\times \int_{t'}^{t} dt_1\ u_a(\mathbf{p};t_1) G_{\alpha\beta}^{(\mathrm{L})}(\mathbf{q};t,t_1)$$

Perturbation expansion with respect to the non-linearity from a Gaussian random state at the infinite past

Velocity



Response function



Velocity equation

$$\frac{\partial}{\partial t} u_i(\mathbf{k};t) = -\nu k^2 u_i(\mathbf{k};t) + iM_{ij\ell}(\mathbf{k}) \iint u_j(\mathbf{p};t) u_\ell(\mathbf{q};t) \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q}$$

We assume that **turbulence is in a Gaussian or random state with high turbulence intensity in the infinite past**, and statistically stationary or quasi-stationary at present.

Turbulence state that is independent of the initial or past condition

Formally integrate this **with regarding the non-linear part as known**

$$u_i(\mathbf{k};t) = v_i(\mathbf{k};t) + iM_{ij\ell}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \times \int_{-\infty}^{t} dt_1 \; g(k;t,t_1) u_j(\mathbf{p};t_1) u_\ell(\mathbf{q};t_1)$$

with $\mathbf{v}(\mathbf{k};t) = \mathbf{A}(\mathbf{k}) \exp\left(-\nu k^2 t\right)$

$g(k;t,t') = \Xi(t-t') \exp\left(-\nu k^2 (t-t')\right)$

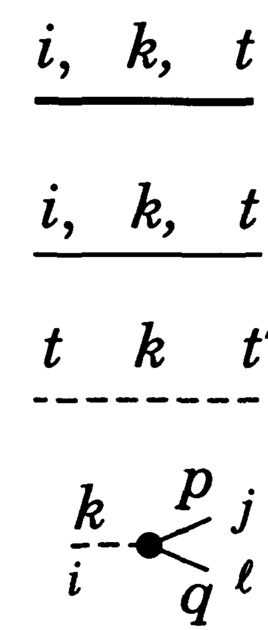
Heaviside step function $\Xi(t) = \begin{cases} 1 & \text{for} \quad t \geq 0, \\ 0 & \text{for} \quad t < 0 \end{cases}$

$u_i(k;t)$:

$v_i(k;t)$:

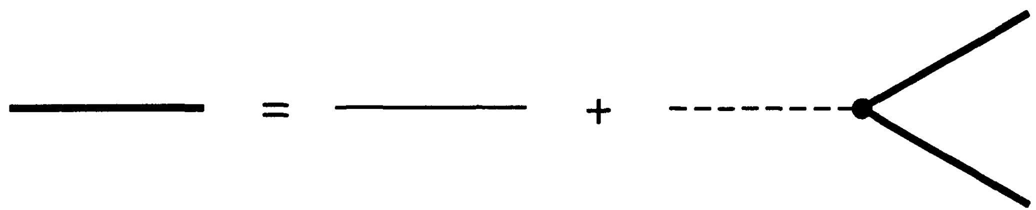
$g(k;t,t')$:

$iM_{ij\ell}(k)$:



$$u_i(\mathbf{k};t) = v_i(\mathbf{k};t) + iM_{ij\ell}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$

$$\times \int_{-\infty}^{t} dt_1 \;\, g(k;t,t_1)u_j(\mathbf{p};t_1)u_\ell(\mathbf{q};t_1)$$

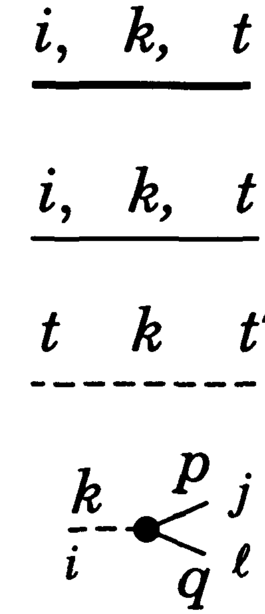


$u_i(k;t)$:

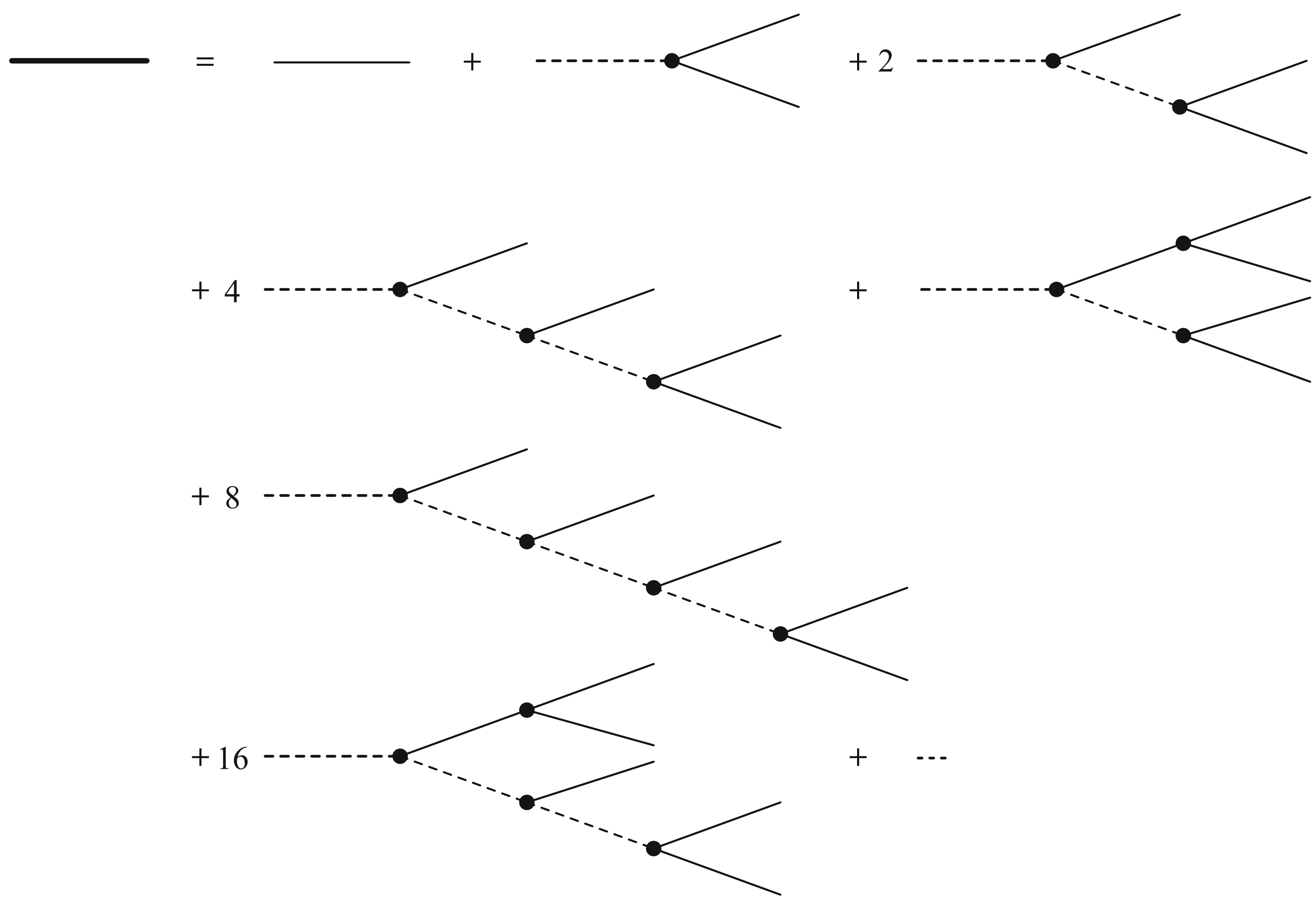
$v_i(k;t)$:

$g(k;t,t')$:

$iM_{ij\ell}(k)$:



Solve this equation in a perturbative manner with **v**(**k**;t) as the leading term:



Response function equation

The response generated by adding an infinitesimal disturbance $\boldsymbol{\delta}\mathbf{f}(\mathbf{k};t)$ to

$$\frac{\partial}{\partial t}u_i(\mathbf{k};t) = -\nu k^2 u_i(\mathbf{k};t) + iM_{ij\ell}(\mathbf{k}) \iint u_j(\mathbf{p};t)u_\ell(\mathbf{q};t)\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$

$$\delta u_i(\mathbf{k};t) = \int d\mathbf{k}' \int_{-\infty}^{t} G'_{ij}(\mathbf{k},\mathbf{k}';t,t_1)\delta f_j(\mathbf{k}';t_1)dt_1$$

The **response function equation** is written as

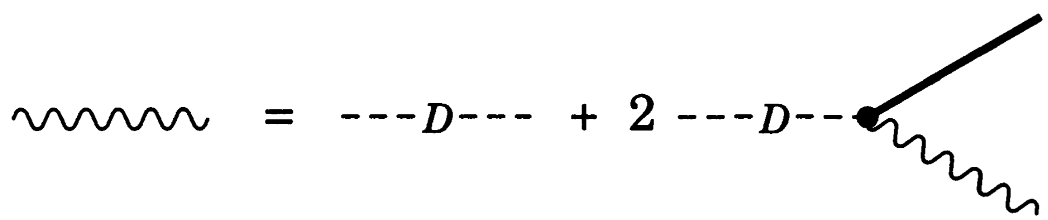
$$\frac{\partial}{\partial t}G'_{ij}(\mathbf{k},\mathbf{k}';t,t') + \nu k^2 G'_{ij}(\mathbf{k},\mathbf{k}';t,t') - D_{ij}(\mathbf{k}')\delta(\mathbf{k}-\mathbf{k}')\delta(t-t') = 2iM_{i\ell m}(\mathbf{k}) \iint u_\ell(\mathbf{p};t)G'_{mj}(\mathbf{q},\mathbf{k}';t,t')\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$

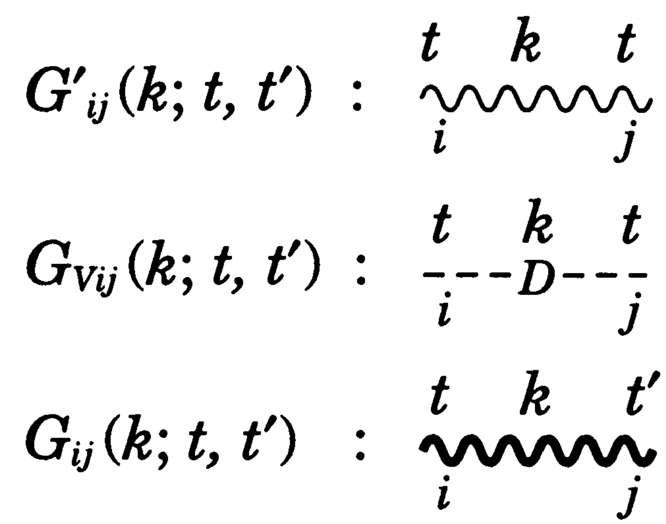
Integrate this with respect to **k**' with putting $G'_{ij}(\mathbf{k},\mathbf{k}';t,t') = G'_{ij}(\mathbf{k};t,t')\delta(\mathbf{k}-\mathbf{k}')$

$$\frac{\partial}{\partial t}G'_{ij}(\mathbf{k};t,t') + \nu k^2 G'_{ij}(\mathbf{k};t,t') - D_{ij}(\mathbf{k})\delta(t-t') = 2iM_{i\ell m}(\mathbf{k}) \iint u_\ell(\mathbf{p};t)G'_{mj}(\mathbf{q};t,t')\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$

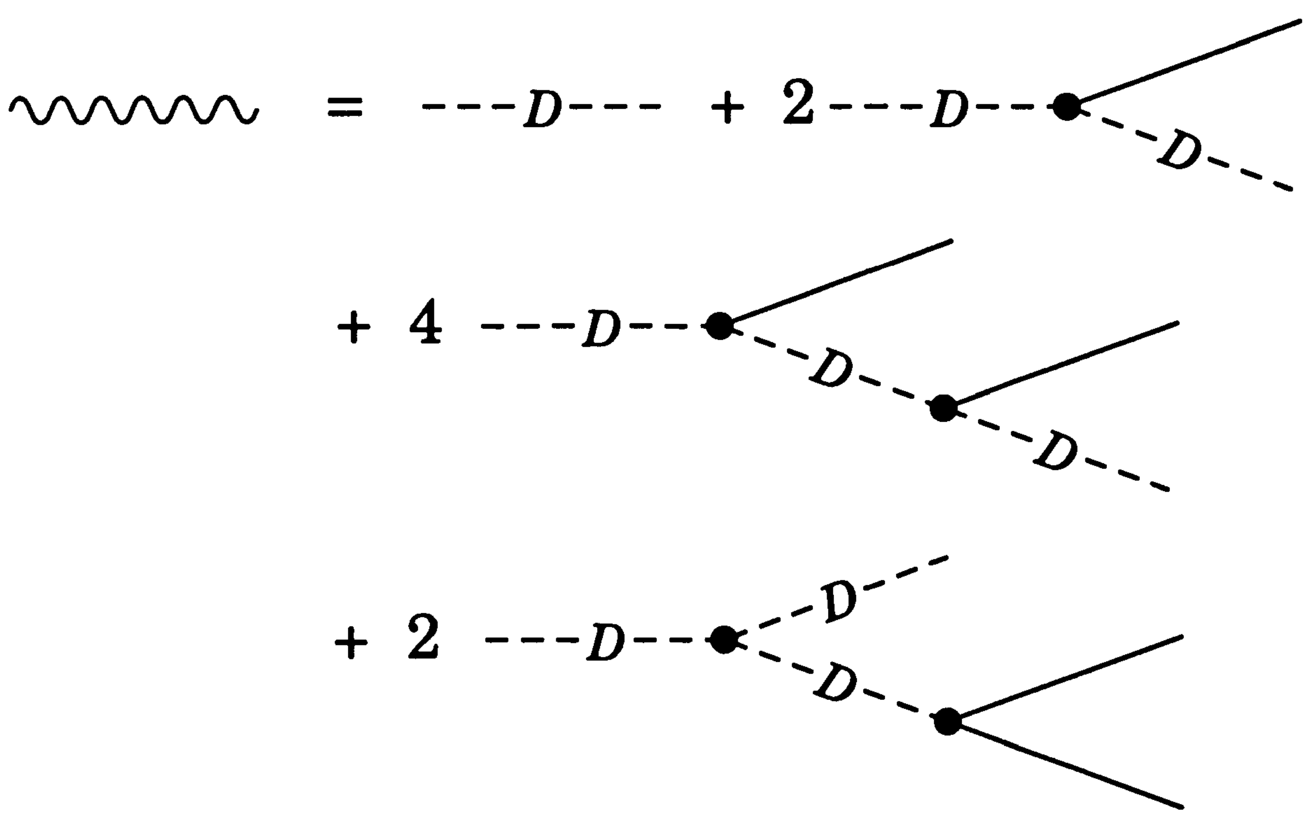
Formally integrate this **with regarding the r.h.s. as known**

$$G'_{ij}(\mathbf{k};t,t') = G_{Vij}(\mathbf{k};t,t') + 2iM_{n\ell m}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$
$$\times \int_{t'}^{t} dt_1 G_{Vin}(\mathbf{k};t,t_1)u_\ell(\mathbf{p};t_1)G'_{mj}(\mathbf{q};t_1,t')$$

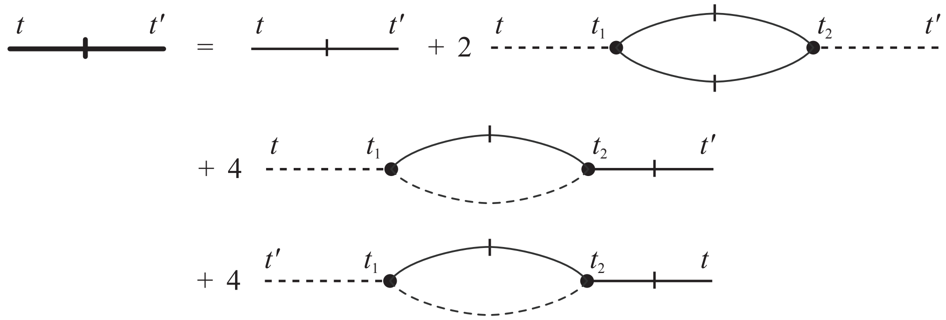




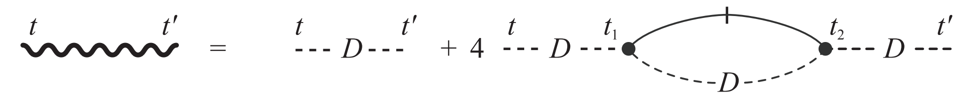
Perturbation expansion up to $O(M^2)$



Correlation function $Q_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')$



Response function $G_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')$



Renormalisation

a part of the infinite series with respect to the propagators is **summed up to the infinite orders**

$$f_{\rm ex}(x) = 1 + x + x^2 + x^3 + \cdots$$

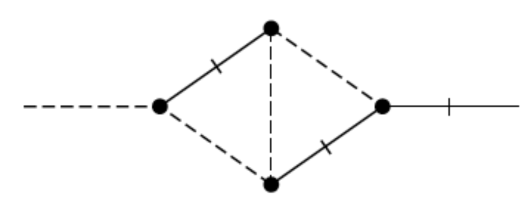
$$f_{\rm ex}(x) = 1 + x(1 + x + x^2 + \cdots)$$

$$f_{\rm ex}(x) = \frac{1}{1-x}$$

No good

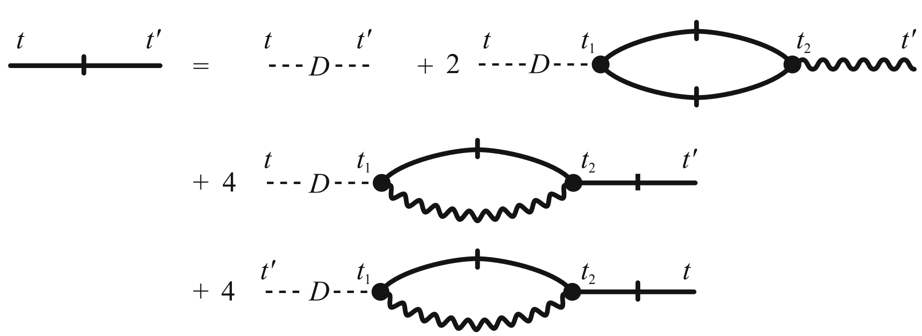
Truncation $f_{\rm ex}(x) = 1 + x(1+x)$

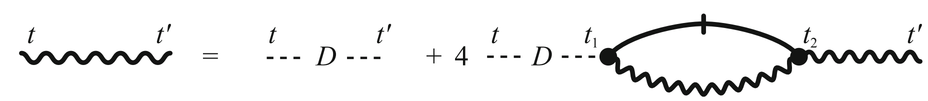
Renormalisation $f_{\rm ex}(x) = 1 + x f_{\rm ex}(x)$



Higher-order vertex not included

DIA = line (propagator) renormalisation (lowest-order in vertex)

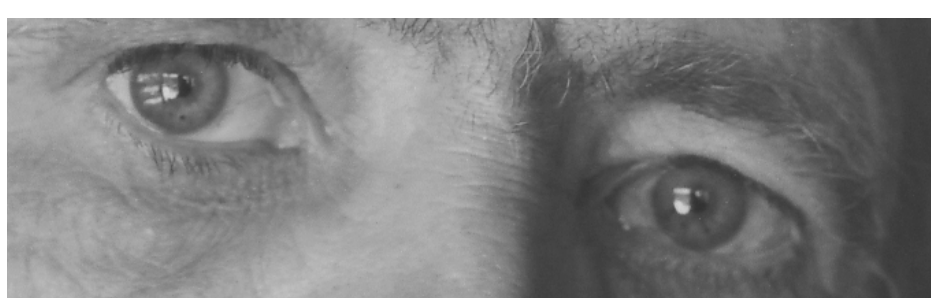




Inhomogeneity, anisotropy, and non-equilibrium properties

RHK with Akira Yoshizawa at IIS in 1996

The difference between the stupidity and genius is that the genius has its limit. (Albert Einstein)



"Crazy"

Multiple-Scale Direct-Interaction Approximation

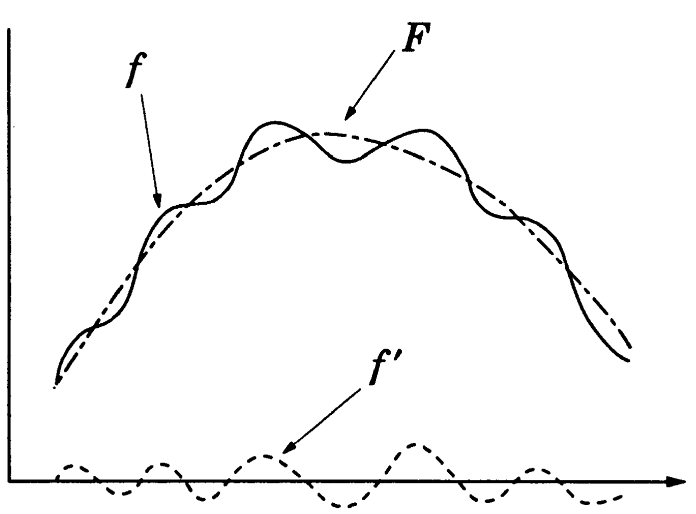
Mirror-symmetric case: Yoshizawa, Phys. Fluids **27**, 1377 (1984)
Non-mirror-symmetric case: Yokoi & Yoshizawa, Phys. Fluids A **5**, 464 (1993)

Fast and slow variables $\boldsymbol{\xi} = \mathbf{x},\ \mathbf{X} = \delta_x \mathbf{x};\ \ \tau = t,\ T = \delta_t t$

Slow variables ***X*** and *T* change only when ***x*** and *t* change much.

$$f = F\left(\mathbf{X};T\right) + f'\left(\boldsymbol{\xi},\mathbf{X};\tau,T\right)$$

$$\nabla = \nabla_{\boldsymbol{\xi}} + \delta_x \nabla_{\mathbf{X}};\quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \delta_t \frac{\partial}{\partial T}$$



Velocity-fluctuation equation $(\delta_x = \delta_t)$

$$\frac{\partial u'_\alpha}{\partial \tau} + U_a \frac{\partial u'_\alpha}{\partial \xi_a} + \frac{\partial}{\partial \xi_a} u'_a u'_\alpha + \frac{\partial p'}{\partial \xi_\alpha} - \nu \nabla_\xi^2 u'_\alpha$$

$$= \delta \left(-u'_a \frac{\partial U_\alpha}{\partial X_a} - \frac{D u'_\alpha}{DT} - \frac{\partial p'}{\partial X_\alpha} - \frac{\partial}{\partial X_a} \left(u'_a u'_\alpha - R_{a\alpha} + 2\nu \frac{\partial^2 u'_\alpha}{\partial X_a \partial \xi_a} \right) \right)$$

$$+\delta^2 \left(\nu \nabla_X^2 u'_\alpha \right)$$

$$\frac{\partial u'_a}{\partial \xi_a} + \delta \frac{\partial u'_a}{\partial X_a} = 0$$

Inhomogeneities, anisotropies, non-equilibrium properties

$$\frac{D}{DT} = \frac{\partial}{\partial T} + \mathbf{U} \cdot \nabla_X$$

Multiple-Scale DIA calculations

Scale parameter expansion $f' = f'_0 + \delta f'_1 + \delta^2 f'_2 + \cdots = \sum_n \delta^n f'_n$

Basic-field (lowest-order field) equation

$$\frac{\partial u_0^i(\mathbf{k};\tau)}{\partial \tau} + \nu k^2 u_0^i(\mathbf{k};\tau) - iM^{ijk}(\mathbf{k}) \iint d\mathbf{p} d\mathbf{q}\ \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) u_0^j(\mathbf{p};\tau) u_0^k(\mathbf{q};\tau) = 0$$

Projection operators

$$M^{ijk}(\mathbf{k}) = \frac{1}{2}\left[k^j D^{ik}(\mathbf{k}) + k^k D^{ij}(\mathbf{k})\right], \quad D^{ij}(\mathbf{k}) = \delta^{ij} - \frac{k^i k^j}{k^2}$$

Green's function

$$\frac{\partial G'_{\alpha\beta}\left(\mathbf{k};\tau,\tau'\right)}{\partial \tau} + \nu k^2 G'_{\alpha\beta}\left(\mathbf{k};\tau,\tau'\right) -2iM^{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}\left(\mathbf{p};\tau\right) G'_{b\beta}\left(\mathbf{q};\tau,\tau'\right) = D_{\alpha\beta}(\mathbf{k}) \delta\left(\tau-\tau'\right)$$

1st-order field

$$\frac{\partial u'_{1\alpha}(\mathbf{k};\tau)}{\partial \tau} + \nu k^2 u'_{1\alpha}(\mathbf{k};\tau)$$

$$-2iM_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau) u'_{S1b}(\mathbf{q};\tau)$$

$$= -D_{\alpha b}(\mathbf{k}) u'_{0a}(\mathbf{k};\tau) \frac{\partial U_b}{\partial X_a} - D_{\alpha a}(\mathbf{k}) \frac{D u'_{0a}(\mathbf{k};\tau)}{D T_{\mathrm{I}}}$$

$$+2M_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{q_b}{q^2} u'_{0a}(\mathbf{p};\tau) \frac{\partial u'_{0c}(\mathbf{q};\tau)}{\partial X_{\mathrm{I}c}}$$

$$-D_{\alpha d}(\mathbf{k}) M_{abcd}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{\partial}{\partial X_{\mathrm{I}c}} \left(u'_{0a}(\mathbf{p};\tau) u'_{0b}(\mathbf{q};\tau) \right)$$

Inhomogeneities, anisotropies, and non-equilibrium properties

Formal solution in terms of Green's function

$$u'_{1\alpha}(\mathbf{k};\tau) = -\frac{\partial U_b}{\partial X_a} \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha b}(\mathbf{k};\tau,\tau_1) u'_{0a}(\mathbf{k};\tau_1)$$

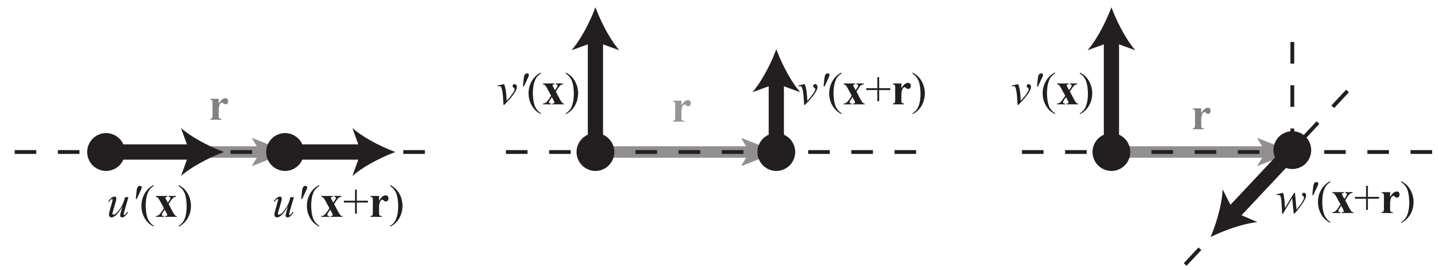
$$- \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha a}(\mathbf{k};\tau,\tau_1) \frac{D u'_{0a}(\mathbf{k};\tau_1)}{D T_1}$$

$$+ 2M_{dab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha d}(\mathbf{k};\tau,\tau_1) \frac{q_b}{q^2} u'_{0a}(\mathbf{p};\tau_1) \frac{\partial u'_{0c}(\mathbf{q};\tau_1)}{\partial X_{\mathrm{I}c}}$$

$$- M_{abcd}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha d}(\mathbf{k};\tau,\tau_1) \frac{\partial}{\partial X_{\mathrm{I}c}} \left(u'_{0a}(\mathbf{p};\tau_1) u'_{0b}(\mathbf{q};\boldsymbol{\tau_1}) \right)$$

Statistical assumptions on the lowest-order (basic) fields

Basic fields: homogeneous isotropic



Longitudinal Transverse Cross

Scalar (isotropic) quantity

Should be invariant under rotation of **r**, **a**, **b**

$$a^i \widetilde{R}^{ij}(\mathbf{r};t,t')b^j = A(\mathbf{r};t,t')\mathbf{a}\cdot\mathbf{b} + B(\mathbf{r};t,t')(\mathbf{r}\cdot\mathbf{a})(\mathbf{r}\cdot\mathbf{b}) + C(\mathbf{r};t,t')\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}=\mathbf{e}^i,\ \ \mathbf{b}=\mathbf{e}^j \longrightarrow \widetilde{R}^{ij}(\mathbf{r};t,t') = A(\mathbf{r};t,t')\delta^{ij} + B(\mathbf{r};t,t')r^i r^j + C(\mathbf{r};t,t')\epsilon^{ij\ell}r^\ell$$

Equivalently, $$\widetilde{R}^{ij}(\mathbf{r};t,t') = f\frac{r^i r^j}{r^2} + g\left(\delta^{ij} - \frac{r^i r^j}{r^2}\right) + h\epsilon^{ij\ell}\frac{r^\ell}{r}$$

$$\frac{\langle {\vartheta'_{\mathrm{B}}}^{\alpha}(\mathbf{k};\tau){\chi'_{\mathrm{B}}}^{\beta}(\mathbf{k}';\tau')\rangle}{\delta(\mathbf{k}+\mathbf{k}')}$$

$$= \Pi^{\alpha\beta}(\mathbf{k})Q_{\vartheta\chi\mathrm{C}}(k;\tau,\tau') + D^{\alpha\beta}(\mathbf{k})Q_{\vartheta\chi\mathrm{S}}(k;\tau,\tau') + \frac{i}{2}\frac{k^c}{k^2}\epsilon^{\alpha\beta c}H_{\vartheta\chi}(k;\tau,\tau')$$

Longitudinal Transverse Cross

with solenoidal and dilatational projection operators

$$D^{\alpha\beta}(\mathbf{k}) = \delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2},\ \ \Pi^{\alpha\beta}(\mathbf{k}) = \frac{k^\alpha k^\beta}{k^2}$$

Calculation of turbulent correlation by DIA

$$\begin{aligned}\langle f'(\mathbf{x};t)g'(\mathbf{x};\mathbf{t})\rangle &= \int d\mathbf{k}\,\langle f'(\mathbf{k};\tau)g'(\mathbf{k};\tau)\rangle\,/\delta(\mathbf{0})\\ &= \int d\mathbf{k}\,\left(\langle f'_0 g'_0\rangle + \langle f'_0 g'_1\rangle + \langle f'_1 g'_0\rangle + \cdots\right)/\delta(\mathbf{0})\end{aligned}$$

Mean-field equations in compressible MHD

Yokoi, N., J. Plasma Phys. **84**, 735840501 & 775840603 (2018a,b)

Density

$$\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho} \mathbf{U}) = -\nabla \cdot \langle \rho' \mathbf{u}' \rangle$$

Turb. mass flux

Momentum

$$\frac{\partial}{\partial t} \overline{\rho} U^\alpha + \frac{\partial}{\partial x^a} \overline{\rho} U^a U^\alpha$$
$$= -(\gamma_0 - 1) \frac{\partial}{\partial x^\alpha} \overline{\rho} Q + \frac{\partial}{\partial x^\alpha} \mu S^{a\alpha} + (\mathbf{J} \times \mathbf{B})^\alpha$$
$$- \frac{\partial}{\partial x^\alpha} \left(\overline{\rho} \langle u'^a u'^\alpha \rangle - \frac{1}{\mu_0} \langle b'^a b'^\alpha \rangle + U^a \langle \rho' u'^\alpha \rangle + U^\alpha \langle \rho' u'^a \rangle \right) + R_U^\alpha$$

Reynolds stress

Turb. Maxwell stress

Turb. mass-energy correl.

Turb. energy flux

Internal energy

$$\frac{\partial}{\partial t} \overline{\rho} Q + \nabla \cdot (\overline{\rho} \mathbf{U} Q) = \nabla \cdot \left(\frac{\kappa}{C_V} \nabla Q \right) - \nabla \cdot (\overline{\rho} \langle q' \mathbf{u}' \rangle + Q \langle \rho' \mathbf{u}' \rangle + \mathbf{U} \langle \rho' q' \rangle)$$
$$- (\gamma_0 - 1) \left(\overline{\rho} Q \nabla \cdot \mathbf{U} + \overline{\rho} \langle q' \nabla \cdot \mathbf{u}' \rangle + Q \langle \rho' \nabla \cdot \mathbf{u}' \rangle \right) + R_Q$$

Turb. energy dilatation

Turb. mass dilatation

Magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B} + \langle \mathbf{u}' \times \mathbf{b}' \rangle) + \eta \nabla^2 \mathbf{B}$$

Turb. electromotive force

Some main results of theoretical analysis

Reynolds and turbulent Maxwell stress

eddy viscosity — inhomogeneous helicity

$$\langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}' \rangle_{\mathrm{D}} = -\nu_{\mathrm{K}} \boldsymbol{\mathcal{S}} + \nu_{\mathrm{M}} \boldsymbol{\mathcal{M}} + \eta_H \nabla H \boldsymbol{\Omega}_* + \cdots$$

D: deviatoric part — cross helicity

mean velocity strain $\boldsymbol{\mathcal{S}} = \nabla \mathbf{U} + (\nabla \mathbf{U})^\dagger$ mean magnetic-field strain $\boldsymbol{\mathcal{M}} = \nabla \mathbf{B} + (\nabla \mathbf{B})^\dagger$

$\boldsymbol{\Omega}_*$: absolute mean vorticity (mean vorticity + rotation)

Turbulent electromotive force

Turb. Mag. Diffusivity — Alpha effect

$$\langle \mathbf{u}' \times \mathbf{b}' \rangle = -(\beta + \zeta) \nabla \times \mathbf{B} + \gamma \nabla \times \mathbf{U} + \alpha \mathbf{B} + (\nabla \zeta) \times \mathbf{B}$$

Cross-helicity effect — Magnetic pumping

$$- \chi_\rho \nabla \overline{\rho} \times \mathbf{B} - \chi_Q \nabla Q \times \mathbf{B} - \chi_D \frac{D\mathbf{U}}{Dt} \times \mathbf{B}$$

Compressibility

Turbulent mass flux

$$\langle \rho' \mathbf{u}' \rangle = -\kappa_{\overline{\rho}} \nabla \overline{\rho} - \kappa_Q \nabla Q - \kappa_D \frac{D\mathbf{U}}{DT} - \kappa_B \mathbf{B}$$

Turbulent internal-energy flux

$$\langle q' \mathbf{u}' \rangle = -\eta_Q \nabla Q - \eta_{\overline{\rho}} \nabla \overline{\rho} - \eta_B \mathbf{B}$$

+ Non-equilibrium effects

Illustrative examples

Differential Rotation

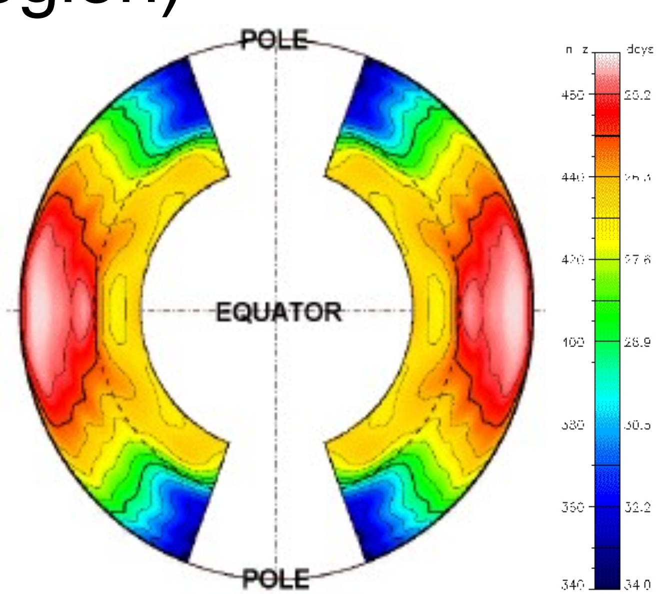
Helioseismology shows the internal structure of the Sun.

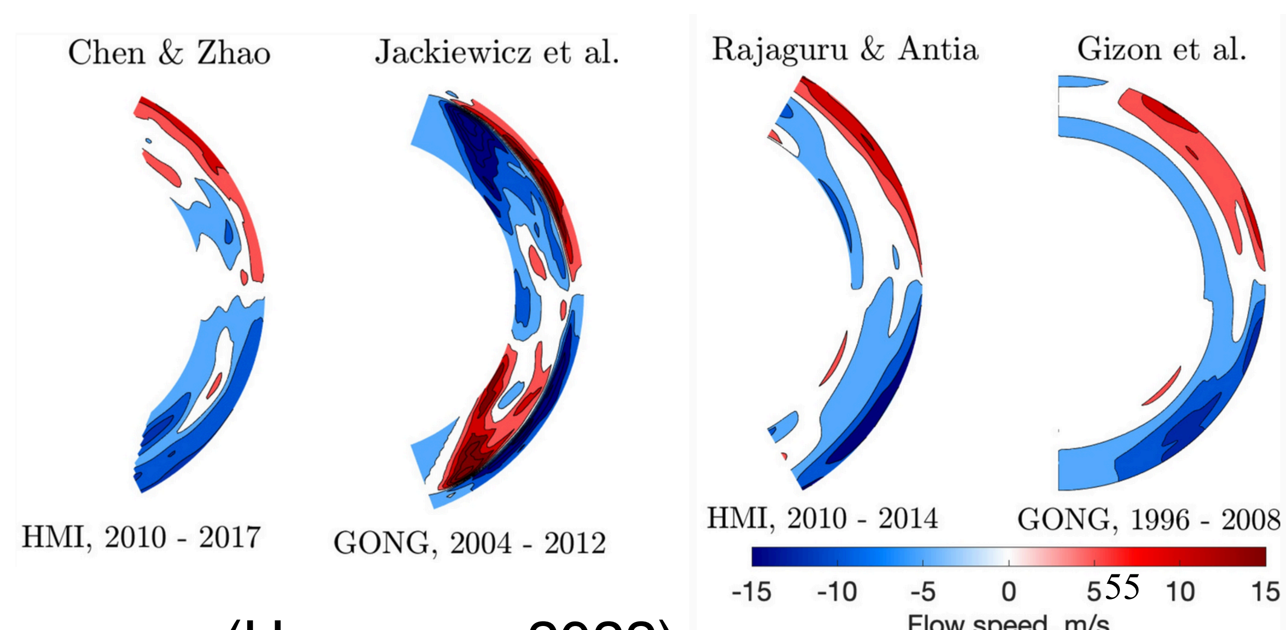
Azimuthal velocity

- Surface **differential rotation** is maintained throughout the convection zone
- Solid body rotation in the radiative interior
- Thin matching zone of shear known as the **tachocline** at the base of the solar convection zone (just in the stable region)

Meridional circulation

- Meridional flow **poleward at surface**
- Interior structure not settled
- **As stars spin faster** they tend to have **slower meridional flows** (from models)



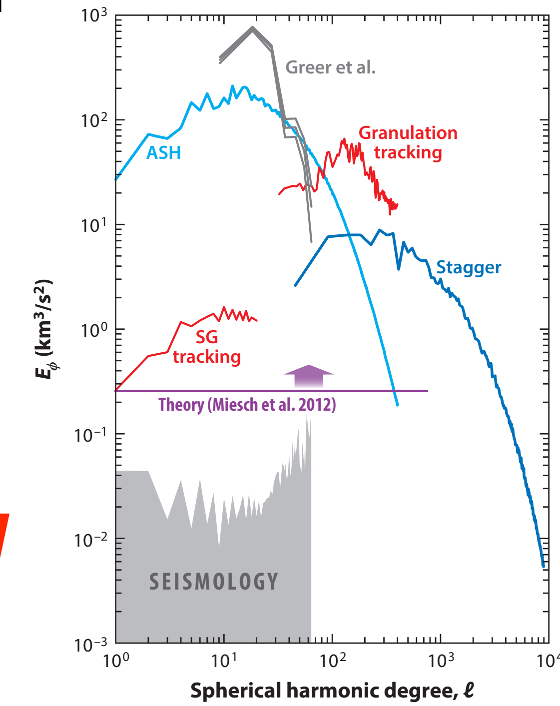


(Hanasoge 2022)

Convection conundrum

The current numerical simulations do not capture some basic characteristics of the solar convection

(Schumacher & Sreenivasan 2020)



(Hanasoge, Gizon & Sreenivasan 2016)

- ***The convective velocity amplitude at large horizontal scales observed by helioseismic investigations is much smaller*** than the one predicted by global convection simulations

- The differential rotation profile obtained by the global numerical simulations shows ***the anti-prograde profile*, if the solar values of luminosity (energy transfer rate) and rotation rate are adopted** in the simulation.

- Also, if the large-scale convection motions are actually small, ***how such weak flows can transfer the solar luminosity and mean differential rotation rate*** observed in the helioseismology?

Approaches to the conundrum

Rotation effect

Amplitude and length-scale of convection could be smaller than we think, leading to a smaller Ro = U/ 2ΩL (Vasil et al 2022, but Käpylä 2023)

-> Inhomogeneous helicity effect coupled with rotation (Yokoi & Yoshizawa 1993, Yokoi & Brandenburg 2016)

Coherent motion effect

Non-local heat transport via entropy rain (e.g., Brandenburg 2016, Anders et al 2019)

-> Non-equilibrium effect associated with plumes and thermals (Yokoi, Masada & Takiwaki 2022)

Magnetic-field effect

Maxwell Stresses from small-scale dynamo acts to counteract Reynolds Stresses that lead to anti-solar rotation profile (e.g., Hotta et al 2022)

-> Cross-helicity effect in angular momentum transport (Yokoi 2023)

Angular-momentum transport by inhomogeneous kinetic helicity

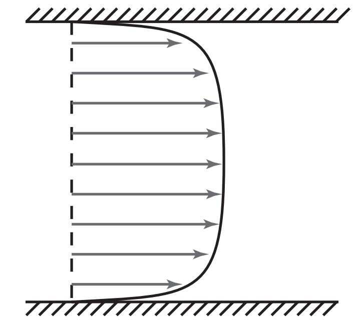
Yokoi, N. & Yoshizawa, A. "Statistical analysis of the effects of helicity in inhomogeneous turbulence," **Phys. Fluids A 5, 464-477 (1993)**
https://doi.org/10.1063/1.858869

Yokoi, N. & Brandenburg, A. "Global flow generation by inhomogeneous helicity," **Phys. Rev. E. 93, 033125-1-14 (2016)**
https://doi.org/10.1103/PhysRevE.93.033125

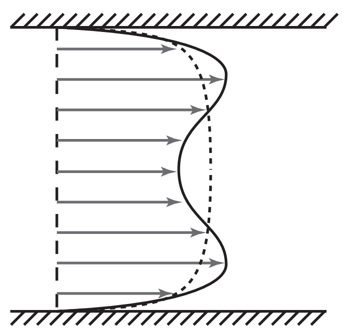
Pouquet, A. & Yokoi, N. "Helical fluid and (Hall)-MHD turbulence: a brief review," **Phil. Trans. Roy. Soc. A 380, 20210081-1-18 (2022)**
https://doi.org/10.1098/rsta.2021.0087

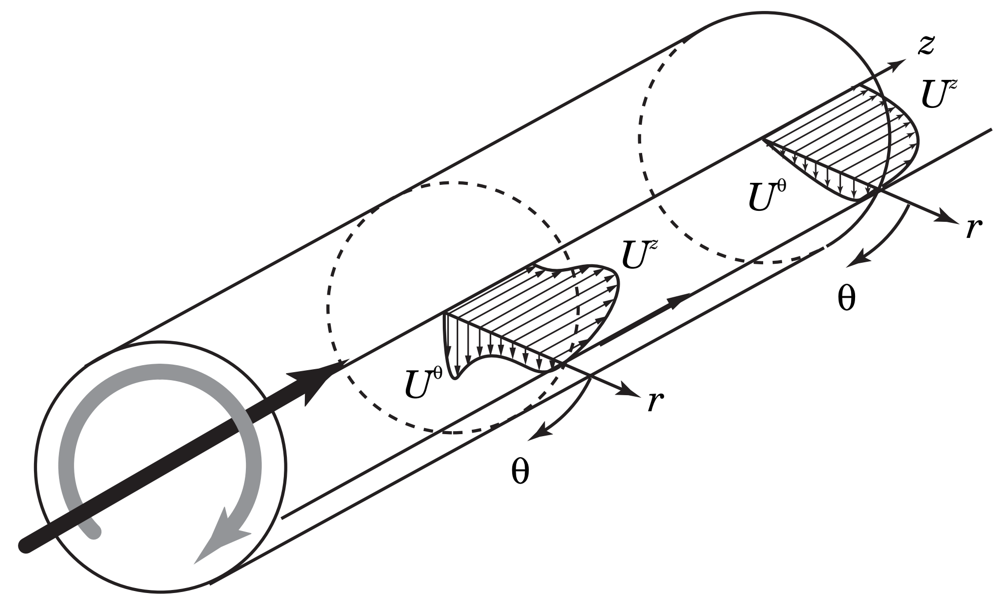
Yokoi, N. "Transports in helical fluid turbulence," pp.25-50,
in Kuzanyan, Yokoi, Georgoulis & Stepanov (eds.) ***AGU Book: Helicities in Geophysics, Astrophysics and Beyond*** **(Wiley, 2023)**
https://doi.org/10.1002/9781119841715

Swirling flow in a circular pipe

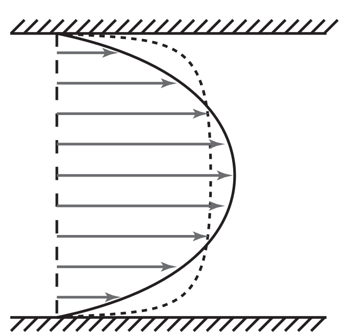


Turbulent swirling pipe flow





Axially rotating turbulent pipe flow



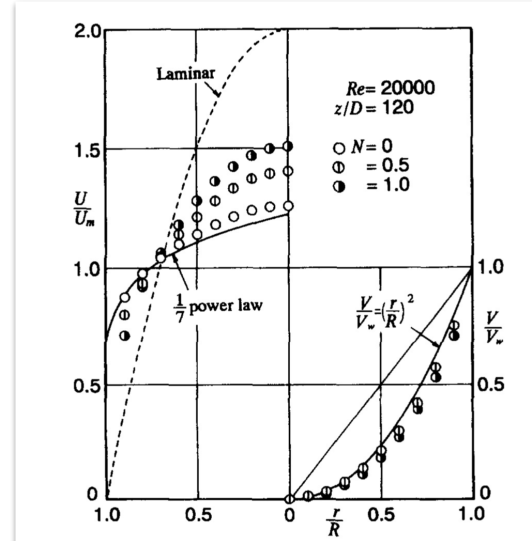


Figure 4 Time-mean velocity profiles (uncertainties in r/R are $\pm 1\%$, and in U/U_m and V/V_w are $\pm 2\%$)

These flow properties cannot be reproduced
by the standard eddy-viscosity representation at all.
Too much dissipative.

(Imao *et al*. 1996)

(Yokoi & Yoshizawa, PoF A**5**, 464, 1993)

Calculation of the Reynolds stress

$$R_{ij} = \langle u'_i(\boldsymbol{\xi}, \mathbf{X}; \tau, T) u'_j(\boldsymbol{\xi}, \mathbf{X}; \tau, T) \rangle = \int R_{ij}(\mathbf{k}, \mathbf{X}; \tau, T) d\mathbf{k}$$

$$\left\langle u'^{\alpha} u'^{\beta} \right\rangle = \left\langle {u'_{\rm B}}^{\alpha} {u'_{\rm B}}^{\beta} \right\rangle + \left\langle {u'_{\rm B}}^{\alpha} {u'_{01}}^{\beta} \right\rangle + \left\langle {u'_{01}}^{\alpha} {u'_{\rm B}}^{\beta} \right\rangle + \cdots$$
$$+ \left\langle {u'_{\rm B}}^{\alpha} {u'_{10}}^{\beta} \right\rangle + \left\langle {u'_{10}}^{\alpha} {u'_{\rm B}}^{\beta} \right\rangle + \cdots$$

$$\left\langle u'^{\alpha} u'^{\beta} \right\rangle_{\rm D} = -\nu_{\rm T} \mathcal{S}^{\alpha\beta} + \left[\Gamma^{\alpha} \left(\Omega^{\beta} + 2\omega_{\rm F}^{\beta} \right) + \Gamma^{\beta} \left(\Omega^{\alpha} + 2\omega_{\rm F}^{\alpha} \right) \right]_{\rm D}$$

where $$\mathcal{S}^{\alpha\beta} = \frac{\partial U^{\alpha}}{\partial x^{\beta}} + \frac{\partial U^{\beta}}{\partial x^{\alpha}} - \frac{2}{3} \nabla \cdot \mathbf{U} \delta^{\alpha\beta}$$

Eddy viscosity $$\nu_{\rm T} = \frac{7}{15} \int {\rm d}\mathbf{k} \int_{-\infty}^{t} d\tau_1 \; G(k; \tau, \tau_1) Q(k; \tau, \tau_1)$$

mixing length $\nu_{\rm T} \sim \tau u^2 \sim u\ell$

Helicity-related coefficient $$\mathbf{\Gamma} = \frac{1}{30} \int k^{-2} {\rm d}\mathbf{k} \int_{-\infty}^{t} d\tau_1 \; G(k; \tau, \tau_1) \nabla H(k; \tau, \tau_1)$$

helicity inhomogeneity is essential

Eddy viscosity + Helicity model

Reynolds stress (Yokoi & Yoshizawa, PoF A**5**, 464, 1993)

$$\mathcal{R}_{\alpha\beta} \equiv \langle u'_\alpha u'_\beta \rangle$$
$$= \frac{2}{3} K \delta_{\alpha\beta} - \nu_{\rm T} \left(\frac{\partial U_\alpha}{\partial x_\beta} + \frac{\partial U_\beta}{\partial x_\alpha} \right) + \eta \left[\Omega_\alpha \frac{\partial H}{\partial x_\beta} + \Omega_\beta \frac{\partial H}{\partial x_\alpha} - \frac{2}{3} \delta_{\alpha\beta} \left(\boldsymbol{\Omega} \cdot \nabla \right) H \right]$$

$$\nu_{\rm T} = C_\nu \tau K, \quad \tau = K/\epsilon, \quad \eta = C_H \tau (K^3/\epsilon^2)$$

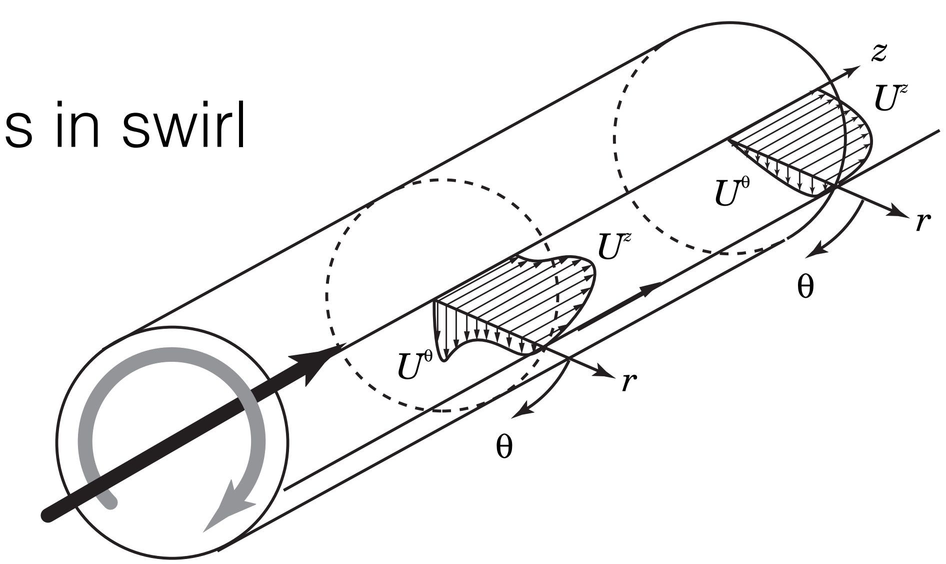
Turbulence quantities

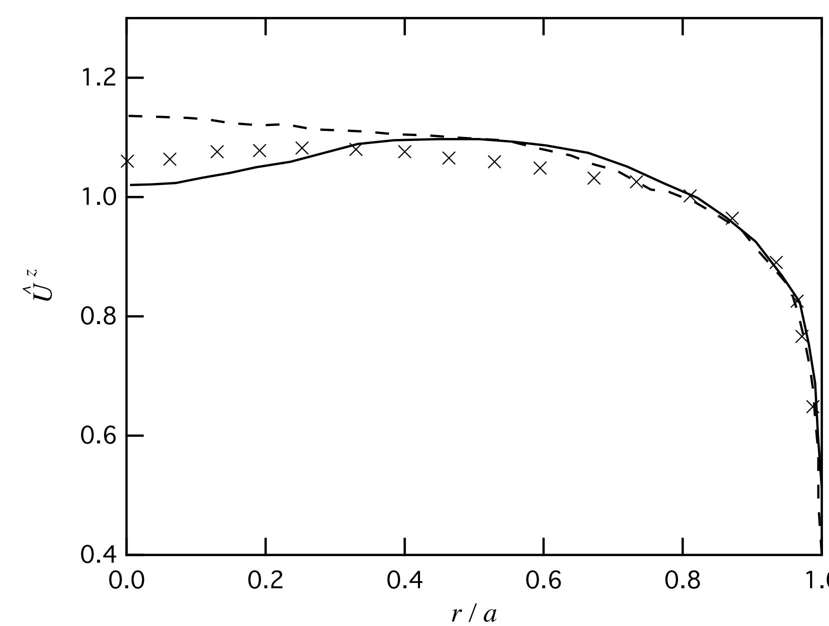
$$K \equiv \frac{1}{2} \langle \mathbf{u}'^2 \rangle, \quad \epsilon \equiv \nu \left\langle \frac{\partial u'_b}{\partial x_a} \frac{\partial u'_b}{\partial x_a} \right\rangle,$$

$$H \equiv \langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle, \quad \epsilon_H \equiv 2\nu \left\langle \frac{\partial u'_b}{\partial x_a} \frac{\partial \omega'_b}{\partial x_a} \right\rangle$$

Helicity turbulence model

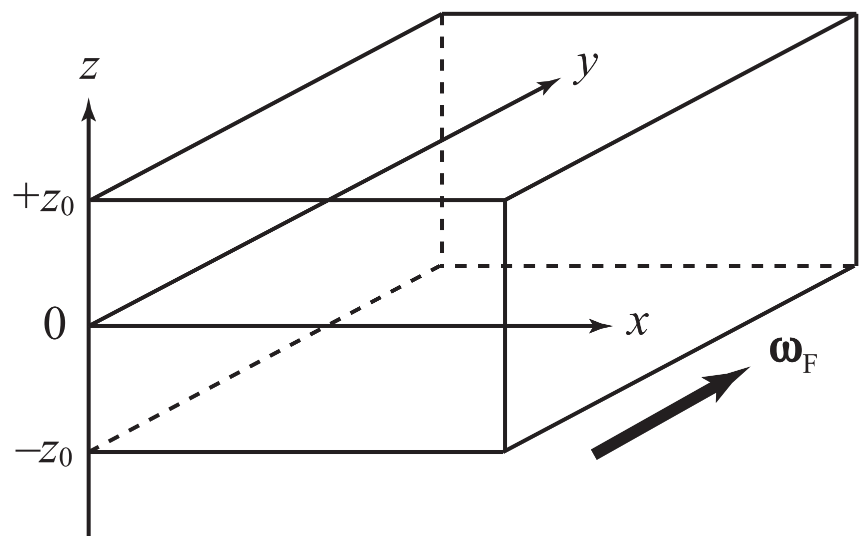
Velocity profiles in swirl

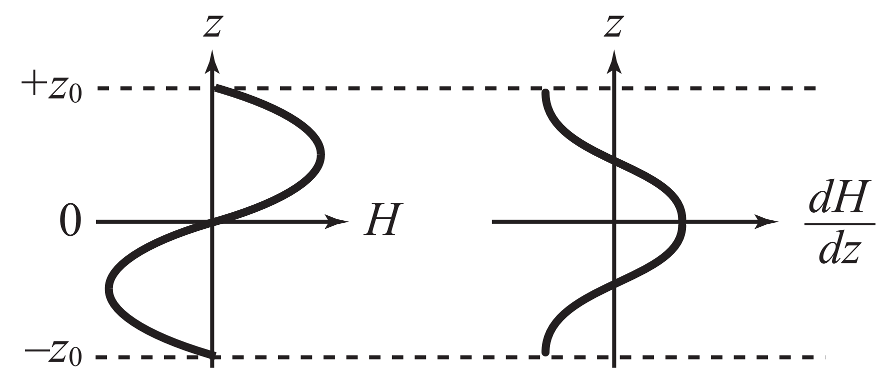




DNS set-up

(Yokoi & Brandenburg, PRE, 2016)





Set-up of the turbulence and rotation $\boldsymbol{\omega}_F$ (left), the schematic spatial profile of the turbulent helicity H (= $\langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle$) (center) and its derivative dH/dz (right).

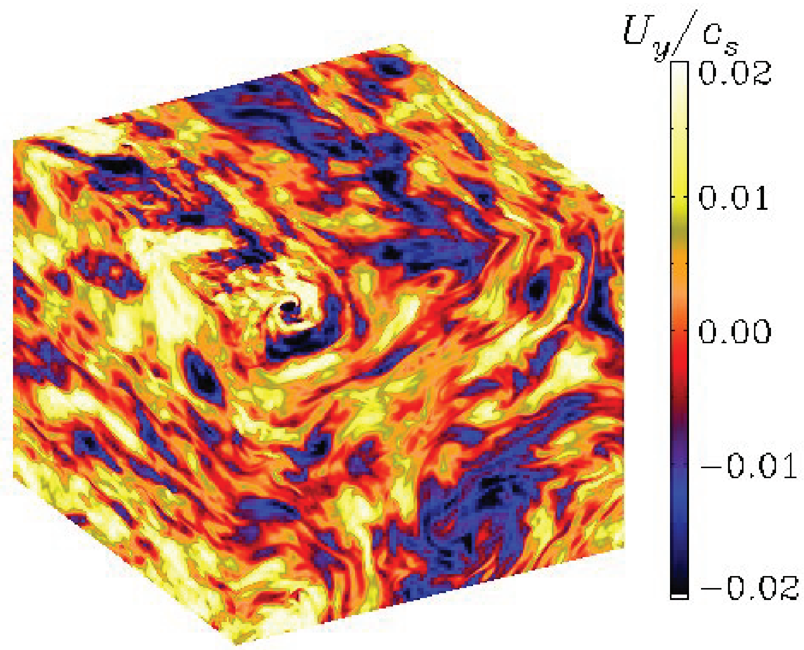
Rotation $$\boldsymbol{\omega}_{\rm F} = (\omega_{\rm F}^x, \omega_{\rm F}^y, \omega_{\rm F}^z) = (0, \omega_{\rm F}, 0)$$

Inhomogeneous turbulent helicity $$H(z) = H_0 \sin(\pi z / z_0)$$

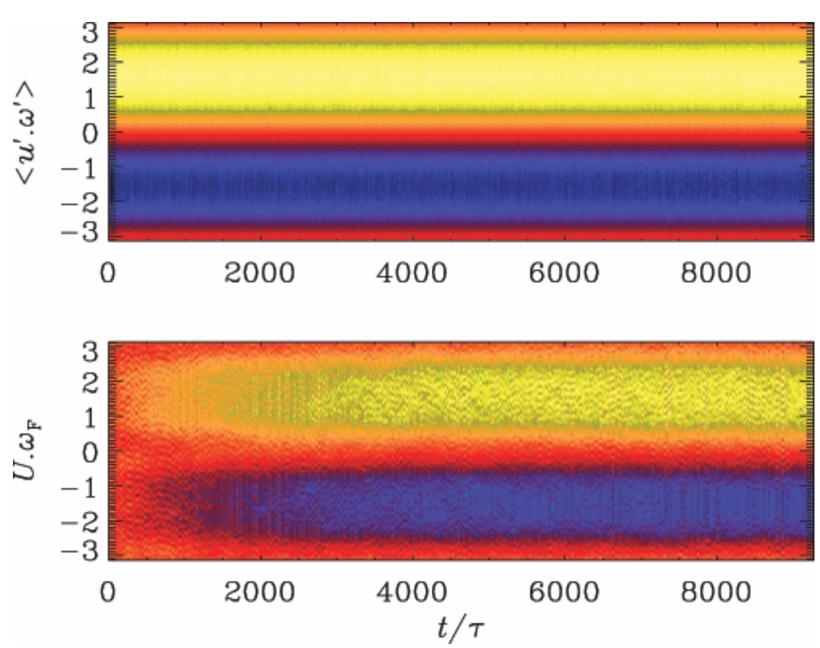
Run	$k_{\rm f}/k_1$	Re	Co	$\eta/(\nu_{\rm T}\tau^2)$
A	15	60	0.74	0.22
B1	5	150	2.6	0.27
B2	5	460	1.7	0.27
B3	5	980	1.6	0.51
C1	30	18	0.63	0.50
C2	30	80	0.55	0.03
C3	30	100	0.46	0.08

Summary of DNS results

Global flow generation



Axial flow component U_y on the periphery of the domain



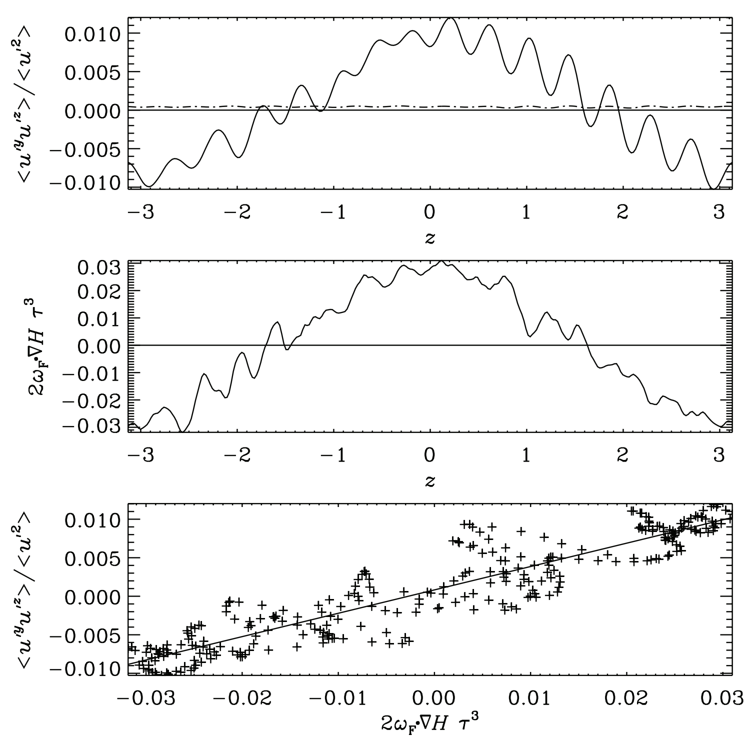
Turbulent helicity $\langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle$ (top) and mean-flow helicity $\mathbf{U} \cdot 2\boldsymbol{\omega}_F$ (bottom)

Reynolds stress

$$\left\langle u'^{\alpha} u'^{\beta} \right\rangle_{\mathrm{D}} = -\nu_{\mathrm{T}} \mathcal{S}^{\alpha\beta} + \left[\Gamma^{\alpha} \left(\Omega^{\beta} + 2\omega_{\mathrm{F}}^{\beta} \right) + \Gamma^{\beta} \left(\Omega^{\alpha} + 2\omega_{\mathrm{F}}^{\alpha} \right) \right]_{\mathrm{D}}$$

Early stage

$$\langle u'^{y} u'^{z} \rangle = \eta 2\omega_{\mathrm{F}}^{y} \frac{\partial H}{\partial z}$$

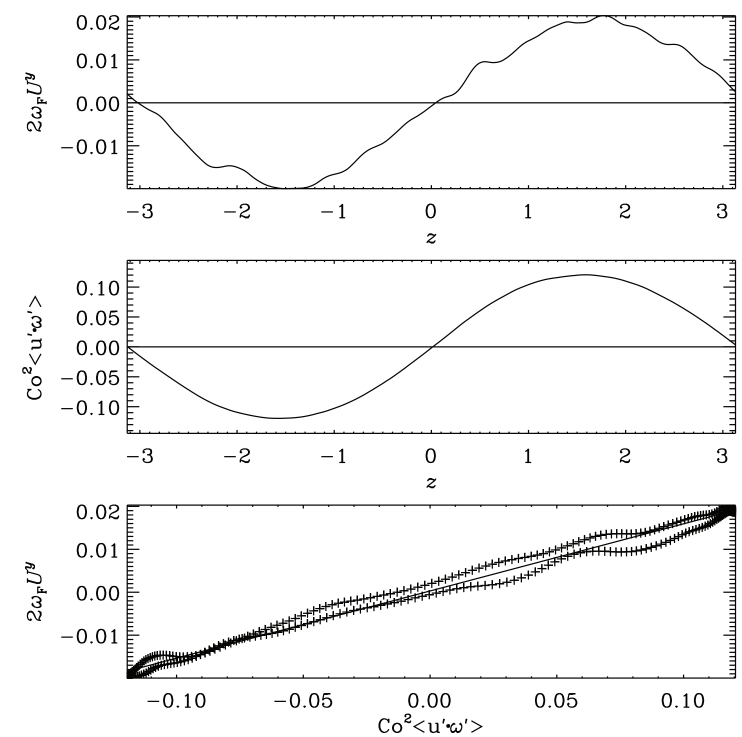


Reynolds stress $\langle u'^{y} u'^{z} \rangle$ (top), helicity-effect term $(\nabla H)^{z}\, 2\omega_{\mathrm{F}}^{y}$ (middle), and their correlation (bottom).

Developed stage

$$\langle u'^{y} u'^{z} \rangle = -\nu_{\mathrm{T}} \frac{\partial U^{y}}{\partial z} + \eta 2\omega_{\mathrm{F}}^{y} \frac{\partial H}{\partial z}$$

$$U^{y} = (\eta / \nu_{\mathrm{T}})\, 2\omega_{\mathrm{F}}^{y} H$$



Mean axial velocity U^{y} (top), turbulent helicity multiplied by rotation $2\omega_{\mathrm{F}} H$ (middle), and their correlation (bottom).

Physical origin

Reynolds stress $\mathcal{R}^{ij} \equiv \langle u'^i u'^j \rangle$

Vortexmotive force $\mathbf{V}_\mathrm{M} \equiv \langle \mathbf{u}' \times \boldsymbol{\omega}' \rangle$

$$V_\mathrm{M}^i = -\frac{\partial \mathcal{R}^{ij}}{\partial x^j} + \frac{\partial K}{\partial x^i}$$

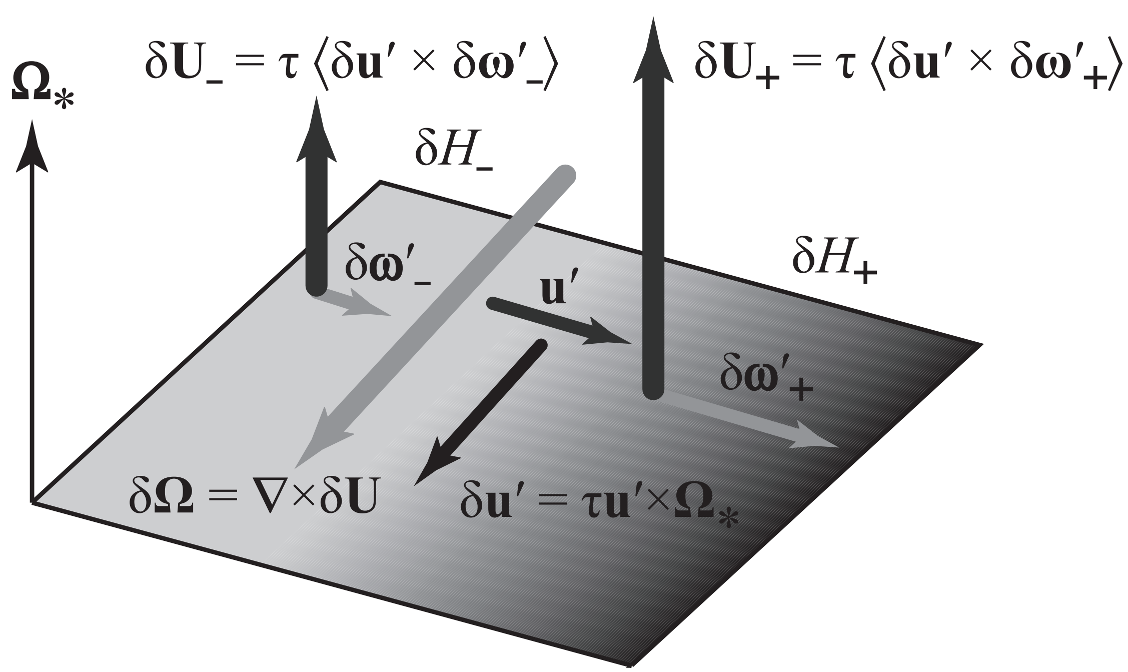
$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nabla \times (\mathbf{U} \times \boldsymbol{\Omega}) + \nabla \times \mathbf{V}_\mathrm{M} + \nu \nabla^2 \boldsymbol{\Omega}$$

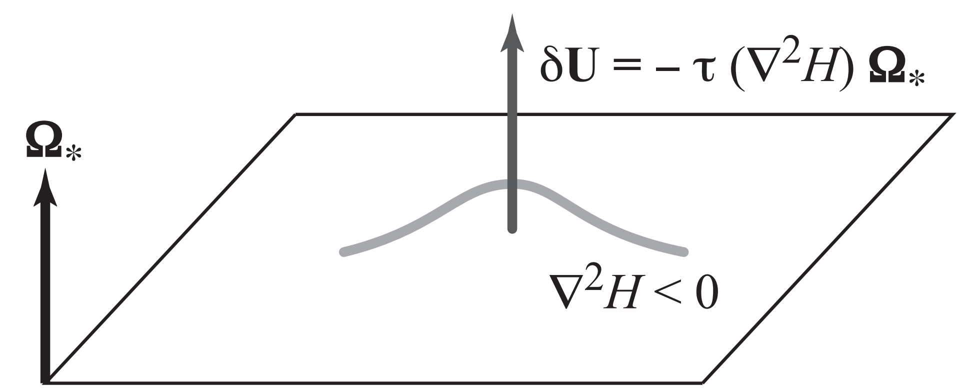
$$\mathbf{V}_\mathrm{M} = -D_\Gamma 2\boldsymbol{\omega}_\mathrm{F} - \nu_\mathrm{T} \nabla \times \boldsymbol{\Omega}$$

$$D_\Gamma = \nabla \cdot \boldsymbol{\Gamma} \propto \nabla^2 H$$

$$\longrightarrow \quad \delta \mathbf{U} \sim -(\nabla^2 H) \boldsymbol{\Omega}_*$$

$$\nabla^2 H \simeq -\frac{\delta H}{\ell^2} = -\frac{\langle \mathbf{u}' \cdot \delta \boldsymbol{\omega}' \rangle}{\ell^2}$$





Mean flow induction consuming eminently localised turbulent helicity

Angular-momentum transport in the solar convection zone

Angular momentum around the rotation axis

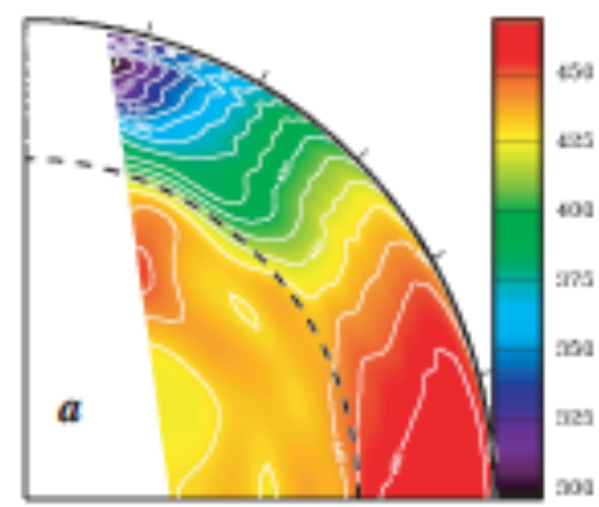
$$L = \Gamma r^2 \omega_{\rm F} + \Gamma r U^\phi \qquad \Gamma = \sin\theta$$

$$\frac{\partial}{\partial t}\rho L + \nabla \cdot (\rho \mathbf{F}_L) = 0$$

Vector flux of angular momentum $\mathbf{F}_L$

$$F_L^r = L U^r + r\Gamma \mathcal{R}^{r\phi}$$

$$F_L^\theta = L U^\theta + r\Gamma \mathcal{R}^{\theta\phi}$$

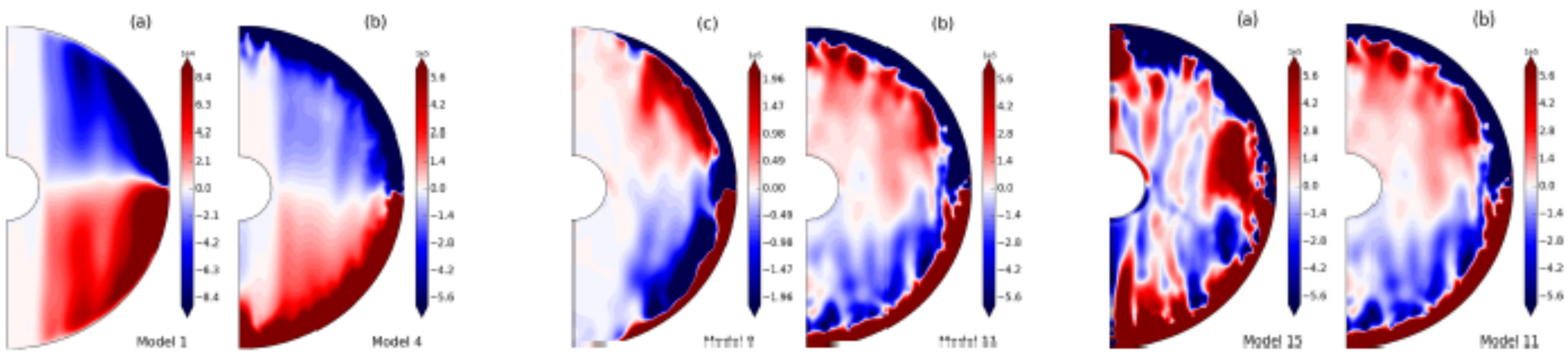


Miesch (2005) Liv. Rev. Sol. Phys. 2005-1

Helicity effect
$$\mathcal{R}_H^{r\phi} = +\frac{\partial H}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r} r U^\theta - \frac{1}{r}\frac{\partial U^r}{\partial \theta}\right)$$

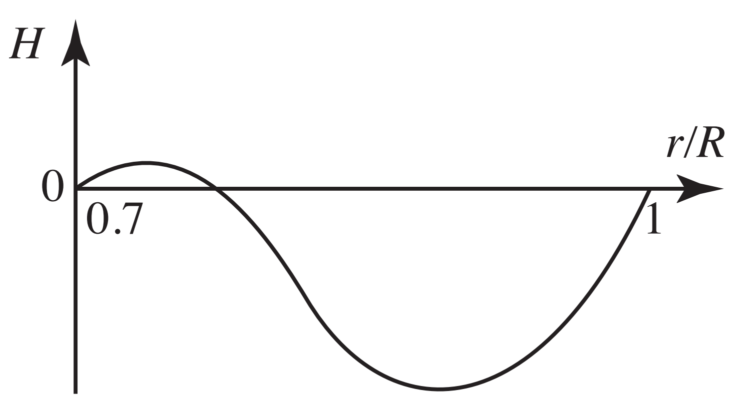
$$\mathcal{R}_H^{\theta\phi} = +\frac{1}{r}\frac{\partial H}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r} r U^\theta - \frac{1}{r}\frac{\partial U^r}{\partial \theta}\right)$$

Helicity effect in the stellar convection zone

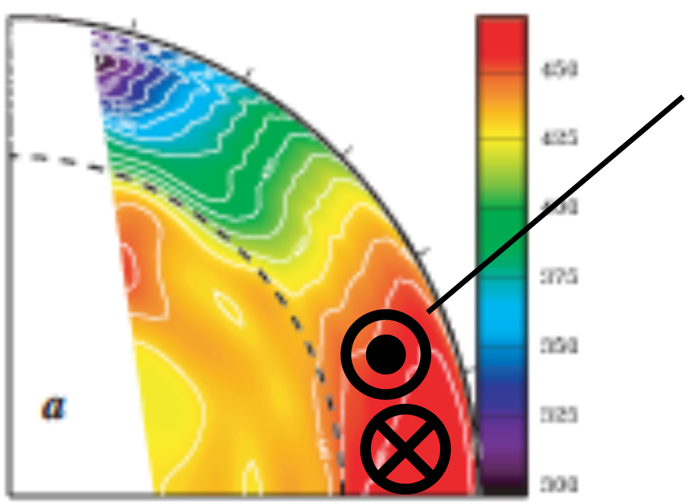


Duarte, et al, (2016) MNRAS **456**, 1708

Schematic helicity distribution



$$\nabla^2 H > 0$$



Mean vorticity due to meridional circulation

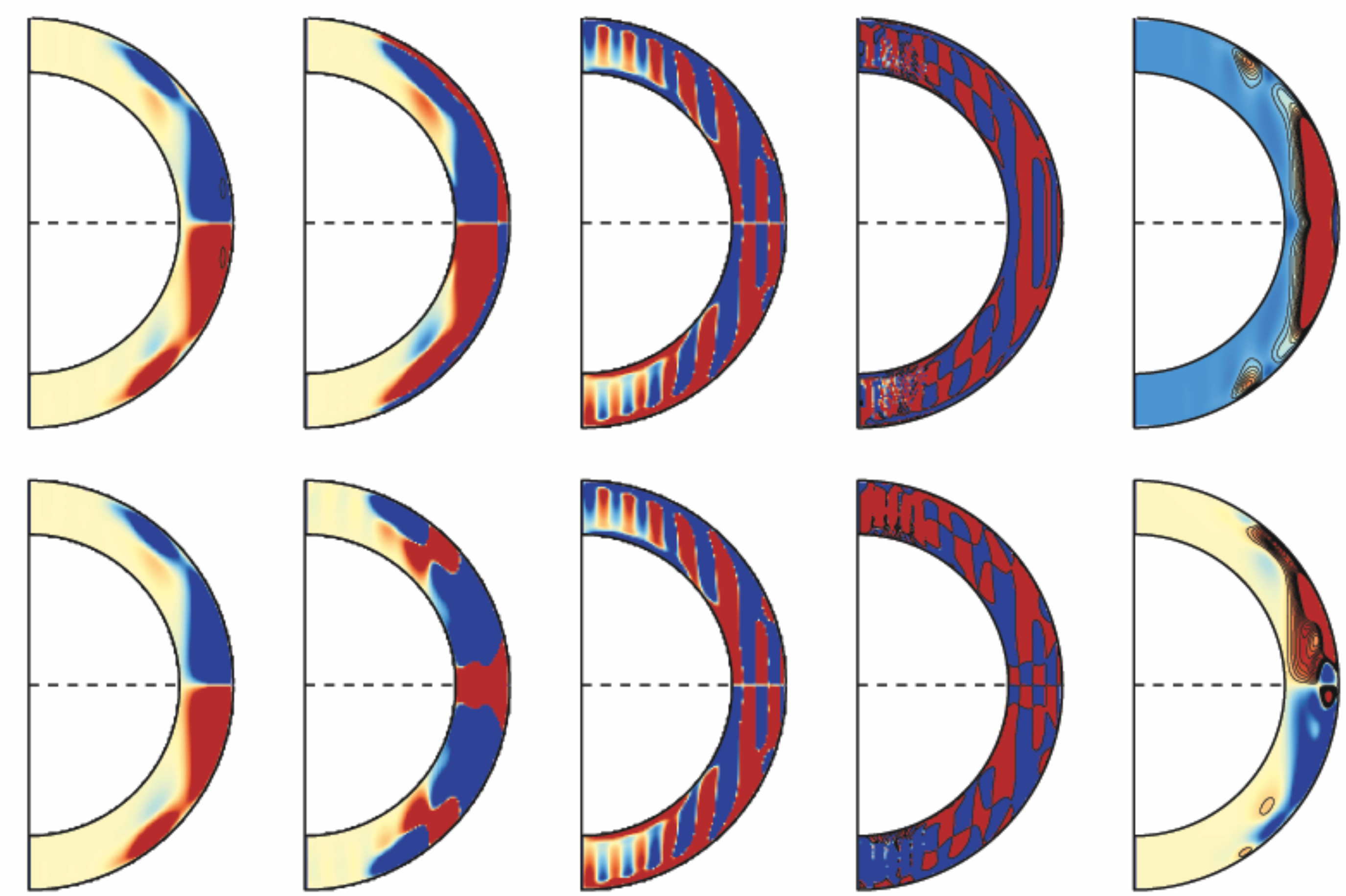
$$\mathbf{\Omega}_{\mathrm{MC}}$$

$$\mathbf{U} \sim -\tau_U (\nabla^2 H) \mathbf{\Omega}_{\mathrm{MC}}$$

Prograde mean velocity induction

Helicity effect in the Reynolds stress

Helicity	Helicity Gradient	Azimuthal Vorticity	Helicity effect	Reynolds stress
$\mathbf{u} \cdot \boldsymbol{\omega}$	$\frac{\partial H}{\partial r}$	$\overline{\Omega}^{\phi}$	$\frac{\partial H}{\partial r} \overline{\Omega}^{\phi}$	$\overline{u'^r u'^\phi}$



$\mathbf{u} \cdot \boldsymbol{\omega} - \overline{\mathbf{u}} \cdot \overline{\boldsymbol{\omega}}$ $(\equiv H)$	$\frac{1}{r} \frac{\partial H}{\partial \theta}$	$\overline{\Omega}^{\phi}$	$\frac{1}{r} \frac{\partial H}{\partial \theta} \overline{\Omega}^{\phi}$	$\overline{u'^\theta u'^\phi}$

(Yokoi & Miesch, GAFD, 2025)

$C_\eta \tau \ell^2 |(\nabla^2 H) \Omega_*|$ Yokoi & Miesch (2025)

Solar parameters

$$v \sim 200 \text{ m s}^{-1} = 2 \times 10^4 \text{cm s}^{-1}$$
$$\ell \sim 200 \text{ Mm} = 2 \times 10^{10} \text{cm}$$
$$\tau \sim \ell / v \sim 10^6 \text{ s}$$

$r\phi$ component

$$\left| \overline{u'^r u'^\phi} \right| \sim 1.2 \times 10^9$$
$$\left| \frac{\partial H}{\partial r} \overline{\Omega}^{\phi} \right| \sim 9.4 \times 10^{-15}$$
$$\tau \ell^2 \left| \frac{\partial H}{\partial r} \overline{\Omega}^{\phi} \right| \sim 10^{12} \rightarrow 10^9$$

with $C_\eta = O(10^{-3})$

$\theta\phi$ component

$$\left| \overline{u'^\theta u'^\phi} \right| \sim 5.6 \times 10^8$$
$$\left| \frac{1}{r} \frac{\partial H}{\partial \theta} \overline{\Omega}^{\phi} \right| \sim 2.6 \times 10^{-15}$$
$$\tau \ell^2 \left| \frac{1}{r} \frac{\partial H}{\partial \theta} \overline{\Omega}^{\phi} \right| \sim 10^{11} \rightarrow 10^8$$

Magnitude same as the Reynolds stress

Validation of the helicity SGS model by DNSs

Yokoi, Mininni, Pouquet, Rosenberg & Marino, Phys. Fluids (to be submitted)

Problem of Constant Adjustment

Smagorinsky model $\nu_{\mathrm{S}} = (C_{\mathrm{S}}\Delta)^2 S$

Smagorinsky constant needs to be adjusted such as

Isotropic flow $C_{\mathrm{S}} \simeq 0.18$

Mixing-layer flow $C_{\mathrm{S}} \simeq 0.15$

Channel flow $C_{\mathrm{S}} \simeq 0.1$

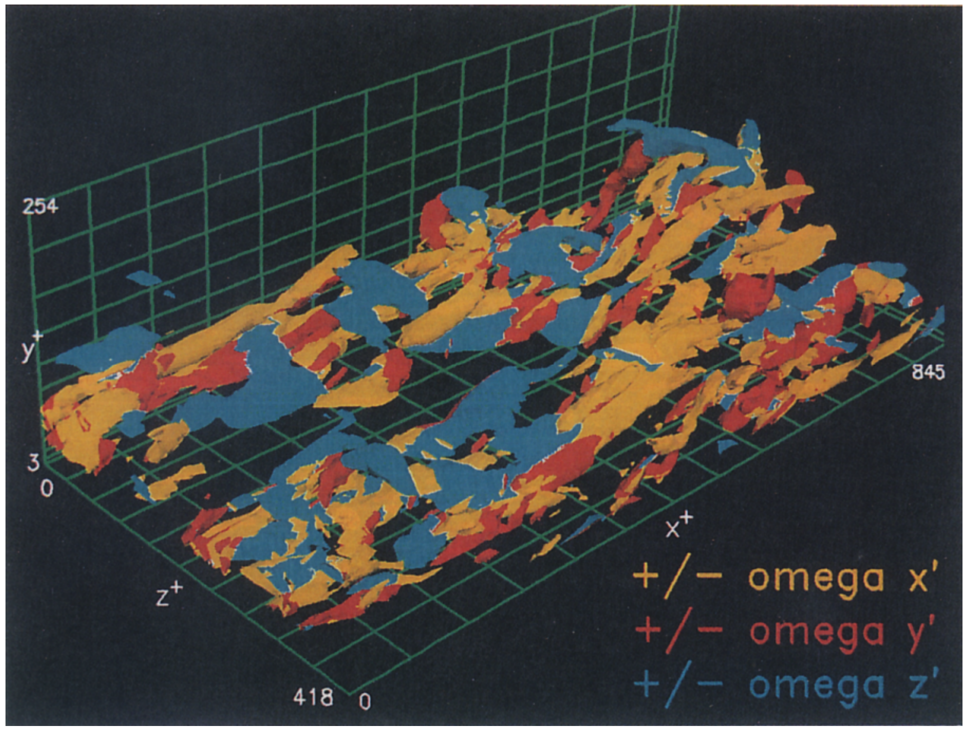
To alleviate

- Dynamic procedure to determine the coefficient C_{S}
- Alternatives to the generic form

$$\mathcal{T}_{\alpha\beta} = C f_{\alpha\beta}\left(\nabla\overline{\mathbf{u}}; \Delta\right) \longrightarrow \mathcal{T}_{\alpha\beta} = C f_{\alpha\beta}\left(\nabla\overline{\mathbf{u}}; \Delta, \cdots\right)$$

- Evolution equations of the SGS quantities

Implication to SGS modelling



Fluctuating vorticity
(Robinson, Kline & Spalart 1988)

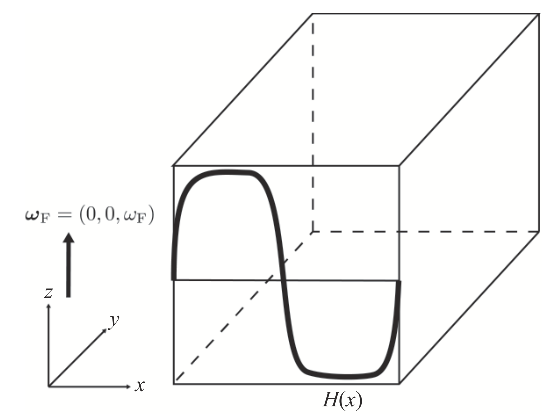
	Smagorinsky constant C_S	Dissipative nature	Vortical structures
Isotropic flow	0.18	more	weak
Mixing-layer flow	0.15	↕	intermediate
Channel flow	0.1	less	strong

Coherent vortical structures (streetwise vorticity)
may be related to the less dissipative nature

SGS stress

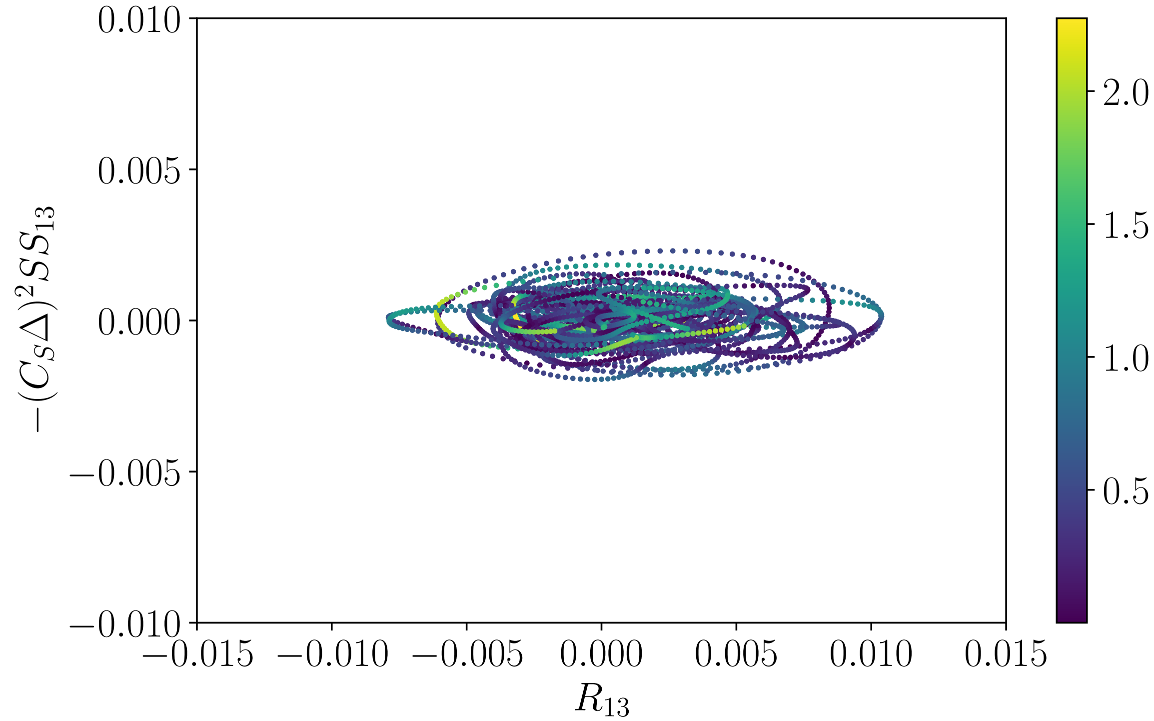
Yokoi, N., Mininni, P., Pouquet, A., et al. Phys. Fluids (2025)

$$\boldsymbol{\tau}_{\rm SD} \equiv \overline{\mathbf{u}\mathbf{u}} - \overline{\mathbf{u}}\,\overline{\mathbf{u}}$$
$$= -\nu_{\rm S}\overline{\boldsymbol{s}} + \eta_{\rm S}\left[\nabla H_{\rm S}\overline{\boldsymbol{\omega}} + (\nabla H_{\rm S}\overline{\boldsymbol{\omega}})^{\dagger}\right]_{\rm D}$$

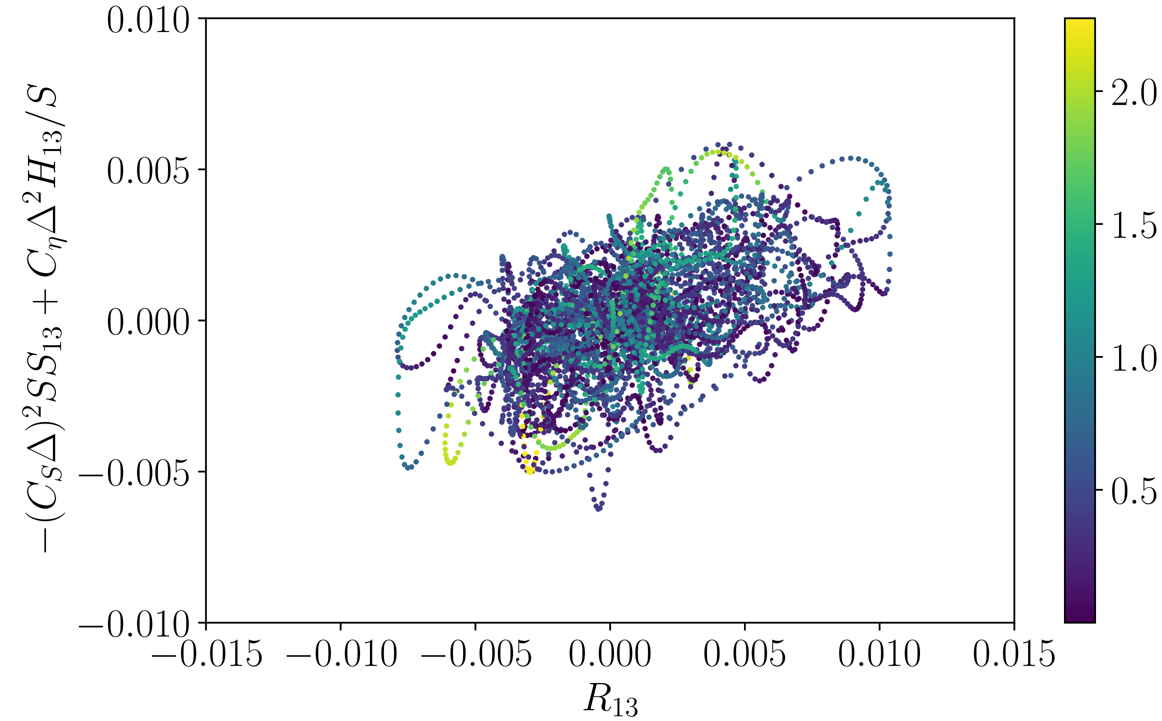


HIT
with a Gaussian filter
at kc=14

Smagorinsky model



Helicity SGS model



$$R_{13} = -(C_{\rm S}\Delta)^2 S S_{13}$$

$$R_{13} = -(C_{\rm S}\Delta)^2 S S_{13} + C_\eta \Delta^2 H_{13}/S$$

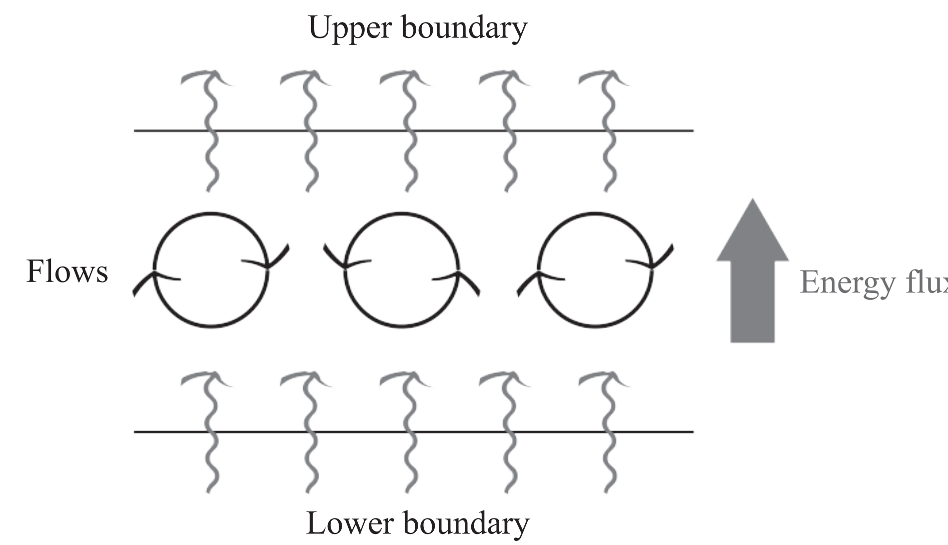
SGS helicity correction improves the SGS stress evaluation

Non-equilibrium effect in convective transport

Yokoi, N., Masada, Y. & Takiwaki, T. “Modelling stellar convective transport with plumes: I. Non-equilibrium turbulence effect in double-averaging formulation,” **Mon. Not. Roy. Astron. Soc. 516, 2718–2735 (2022)**
https://doi.org/10.1093/mnras/stac1181

Yokoi, N. “Non-Equilibrium Turbulent Transport in Convective Plumes Obtained from Closure Theory,” **Atmosphere 14, 1013-1-22 (2023)**
https://doi.org/10.3390/atmos14061013

Non-equilibrium properties



Non-equilibrium open system

Non-equilibrium state is sustained by the energy flux through the boundaries with mass and flow

Deviation from the local equilibrium

Kolmogorov homogeneous isotropic turbulence:
Local equilibrium between the energy production/injection and dissipation

Non-equilibrium property due to the time variation of fluctuations along the large- or meso-scale flows

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \delta \frac{\partial}{\partial T} \longrightarrow \frac{D\mathbf{u}'}{DT} = \frac{\partial \mathbf{u}'}{\partial T} + (\mathbf{U} \cdot \nabla_{\mathbf{X}})\mathbf{u}'$$

Irreversibility Asymmetry with respect to the exchange of time variables

$$H_{ub}(\tau, \tau_1) \neq H_{ub}(\tau_1, \tau)$$

Plumes in stellar convection

Plumes and turbulent transport

Entrainment with plumes

Turner 1973 Importance of the non-equilibrium effects

Linden 2000 Entrainment assumption and scaling

Surface cooling diving plumes

Spruit 1997 Dominant role of the cooling driven plumes

Rieutord & Zhan 1995 Long-lasting effect of diving plumes

Rast 1998 Beyond hydrostatic pressure and entrainment assumption

Two layer polytropic gas configuration

Cossette & Rast 2016 More effective mixing due to radiative cooling

Entropy rain

Brandenburg 2016 Turbulence modelling implementing plume effects

Entrainment model in the supernovae explosion

Murphy & Meakin 2011 Usefulness of entrainment model

Anett + 2015 Chaotic model with roll of Lorentz

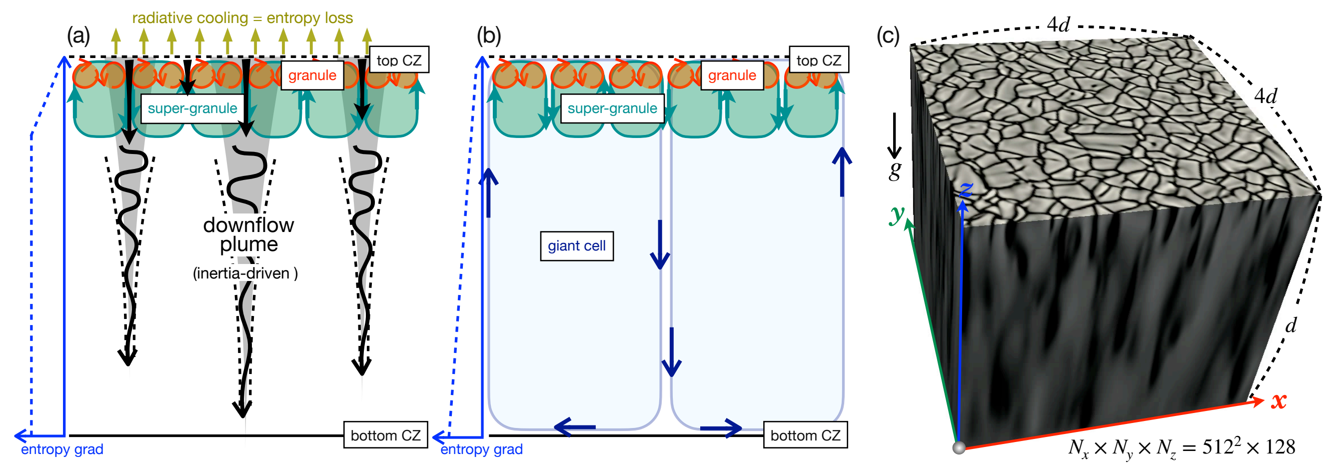
Setup for Stellar Convection

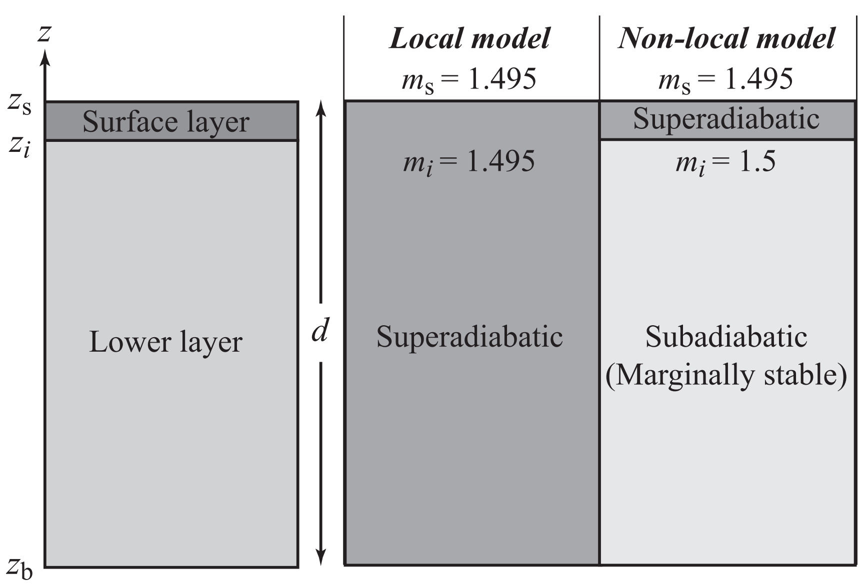
Two-layer polytropic gas configurations

Locally driven case

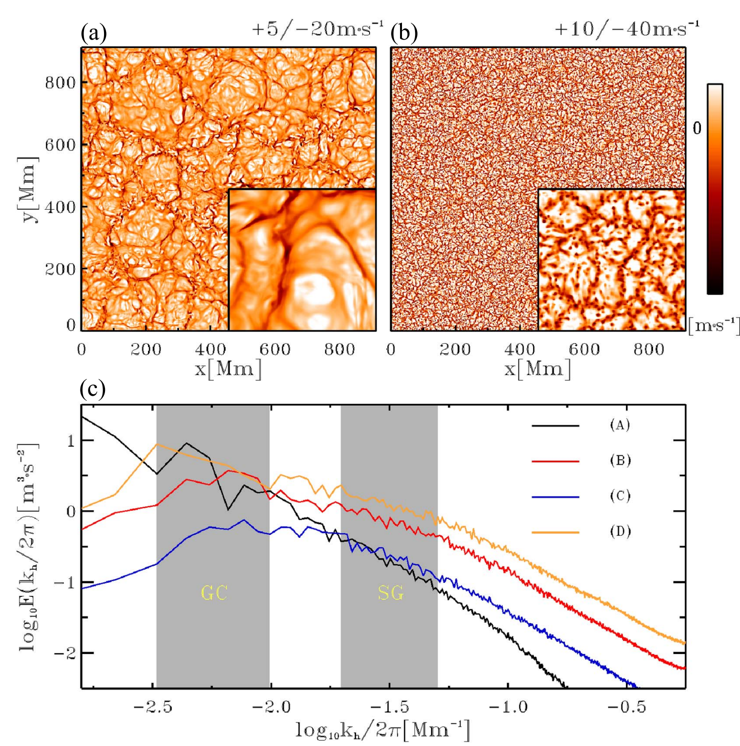
Surface cooling diving plume case

(Cossette & Rast 2016, ApJL, 829, L17)

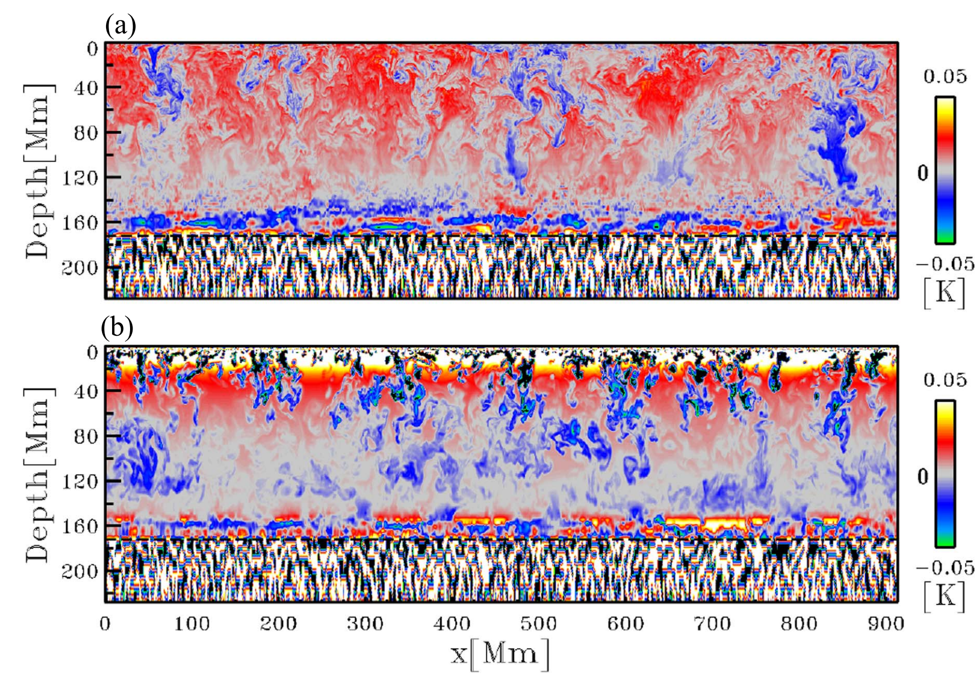




Horizontal cross section of instantaneous vertical velocity and horizontal velocity power spectra

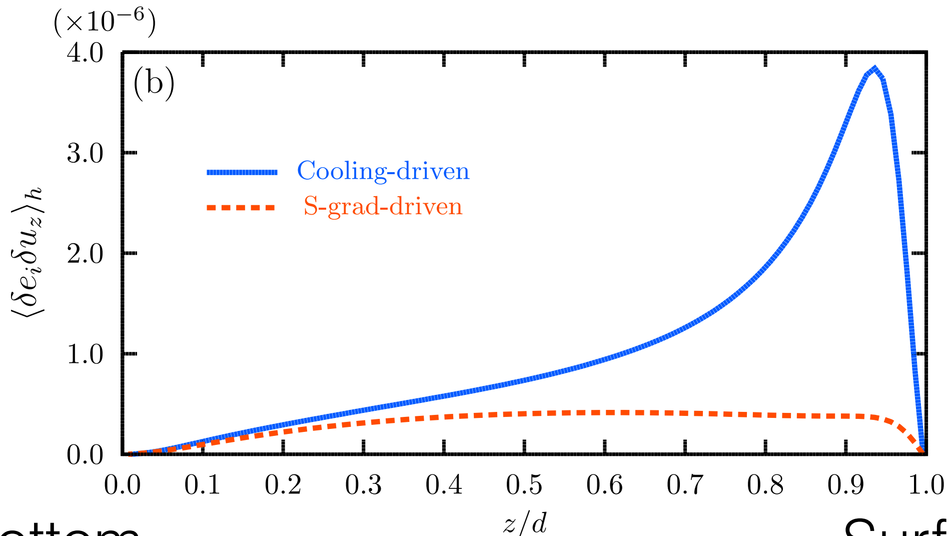


Vertical cross section of instantaneous temperature fluctuations



Spatial distributions of turbulent energy flux $\langle e' u'^z \rangle$

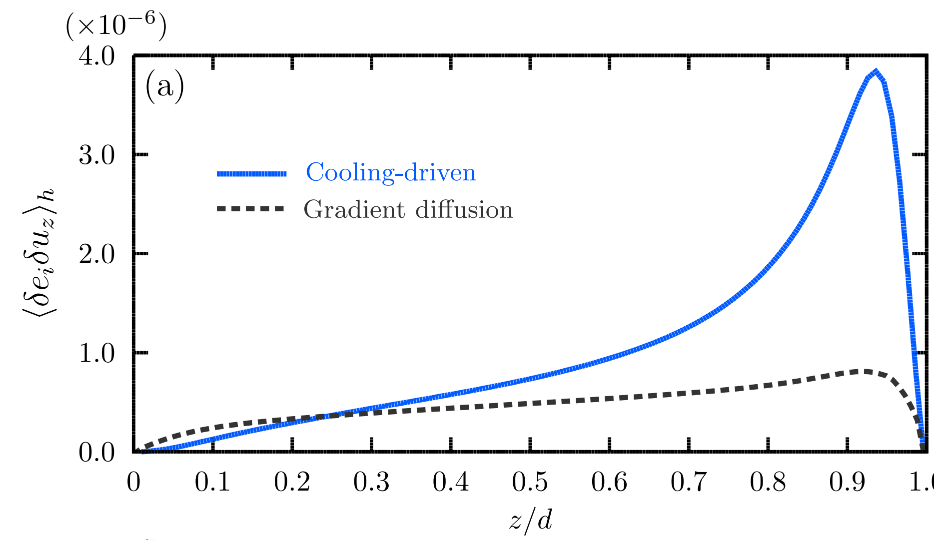
DNS



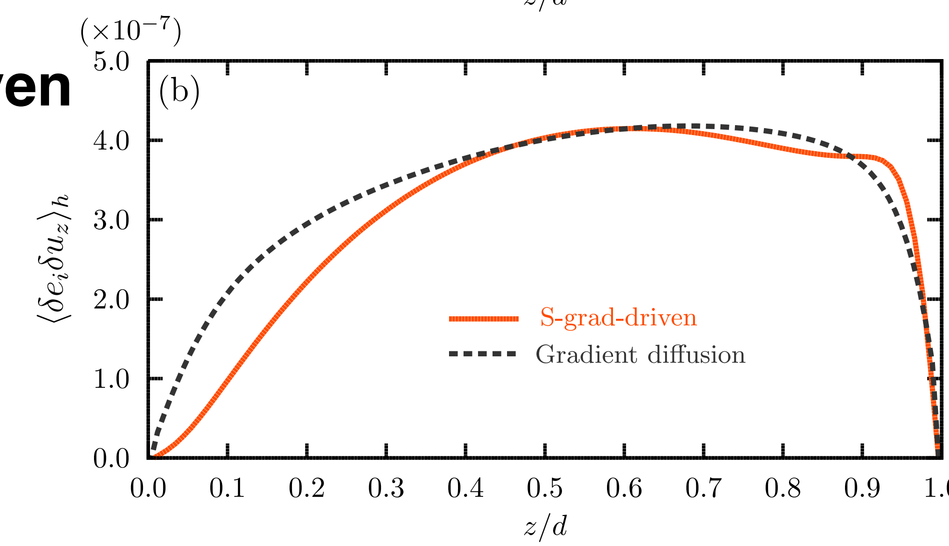
Bottom Surface

Gradient diffusion model with mixing-length theory (MLT)

Non-locally-driven



Locally-driven



Bottom Surface

Turbulent transport of the surface cooling diving plume (non-local transport) cannot be properly described by *the gradient-diffusion model with MLT*.

Non-equilibrium effects

From the Multiple-scale DIA calculations

1st-order field

$$
\begin{aligned}
&\frac{\partial u'_{1\alpha}(\mathbf{k};\tau)}{\partial \tau} + \nu k^2 u'_{1\alpha}(\mathbf{k};\tau) \\
&\quad -2iM_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau) u'_{S1b}(\mathbf{q};\tau) \\
= &\quad -D_{\alpha b}(\mathbf{k}) u'_{0a}(\mathbf{k};\tau) \frac{\partial U_b}{\partial X_a} - D_{\alpha a}(\mathbf{k}) \frac{D u'_{0a}(\mathbf{k};\tau)}{D T_{\mathrm{I}}} \\
&\quad +2M_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{q_b}{q^2} u'_{0a}(\mathbf{p};\tau) \frac{\partial u'_{0c}(\mathbf{q};\tau)}{\partial X_{\mathrm{I}c}} \\
&\quad -D_{\alpha d}(\mathbf{k}) M_{abcd}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{\partial}{\partial X_{\mathrm{I}c}} \left(u'_{0a}(\mathbf{p};\tau) u'_{0b}(\mathbf{q};\tau) \right)
\end{aligned}
$$

Formal solution in terms of the response function G

$$
\begin{aligned}
u'^{\alpha}_1(\mathbf{k};\tau) = &- \frac{\partial U^b}{\partial X^a} \int_{-\infty}^{\tau} d\tau_1 G'^{\alpha b}(\mathbf{k};\tau,\tau_1) u'^{a}_0(\mathbf{k};\tau_1) \\
&- \int_{-\infty}^{\tau} d\tau_1 G'^{\alpha a}(\mathbf{k};\tau,\tau_1) \frac{D u'^{a}_0(\mathbf{k};\tau_1)}{D T_{\mathrm{I}}} \\
&+ 2M^{dab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \int_{-\infty}^{\tau} d\tau_1 G'^{\alpha d}(\mathbf{k};\tau,\tau_1) \\
&\qquad \times \frac{q^b}{q^2} u'^{a}_0(\mathbf{p};\tau_1) \frac{\partial u'^{c}_0(\mathbf{q};\tau_1)}{\partial X^c_{\mathrm{I}}} \\
&- M^{abcd}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \int_{-\infty}^{\tau} d\tau_1 G'^{\alpha d}(\mathbf{k};\tau,\tau_1) \\
&\qquad \times \frac{\partial}{\partial X^c_{\mathrm{I}}} \left(u'^{a}_0(\mathbf{p};\tau_1) u'^{b}_0(\mathbf{q};\tau_1) \right)
\end{aligned}
$$

Non-equilibrium Effects

Turbulent energy $K \equiv \langle \mathbf{u}'^2 \rangle /2 = \frac{1}{2} \int d\mathbf{k} \, \langle u'^{\ell}(\mathbf{k};\tau) u'^{\ell}(\mathbf{k}';\tau) \rangle / \delta(\mathbf{k}+\mathbf{k}')$

$$\langle u'^{\ell} u'^{\ell} \rangle = \langle u_0'^{\ell} u_0'^{\ell} \rangle + \delta \langle u_1'^{\ell} u_0'^{\ell} \rangle + \delta \langle u_0'^{\ell} u_1'^{\ell} \rangle + \cdots$$

Length scale (energy containing scale) ℓ_{C} Dissipation rate $\varepsilon = \nu \left\langle \left(\frac{\partial u'^{\ell}}{\partial x^{\ell}} \right)^2 \right\rangle$

$$K = C_{K1} \varepsilon^{2/3} \ell_{\mathrm{C}}^{2/3} - C_{K2} \varepsilon^{-3/2} \ell_{\mathrm{C}}^{4/3} \frac{D\varepsilon}{Dt} - C_{K3} \varepsilon^{1/3} \ell_{\mathrm{C}}^{1/3} \frac{D\ell_{\mathrm{C}}}{Dt}$$

Equilibrium effect **Non-equilibrium effect**

Solve this by iterations with respect to ℓ_{c}

$$\ell_{\mathrm{C}} = C_{\ell 1} K^{3/2} \varepsilon^{-1} + C_{\ell 2} K^{3/2} \varepsilon^{-2} \frac{DK}{Dt} - C_{\ell 3} K^{5/2} \varepsilon^{-3} \frac{D\varepsilon}{Dt}$$

$$\ell_{\mathrm{C}} = \ell_{\mathrm{E}} \left(1 - C'_{\mathrm{N}} \frac{1}{K} \frac{D}{Dt} \frac{K^2}{\varepsilon} \right)$$

Equilibrium length scale $\ell_{\mathrm{E}} = K^{3/2}/\varepsilon$

$$\nu_{\mathrm{T}} = \begin{cases} \nu_{\mathrm{TE}} \left(1 + C_{\mathrm{N}} \frac{1}{K} \frac{D}{Dt} \frac{K^2}{\varepsilon} \right)^{-1} & \text{for} \quad \frac{D}{Dt} \frac{K^2}{\varepsilon} > 0 \\ \nu_{\mathrm{TE}} \left(1 - C_{\mathrm{N}} \frac{1}{K} \frac{D}{Dt} \frac{K^2}{\varepsilon} \right) & \text{for} \quad \frac{D}{Dt} \frac{K^2}{\varepsilon} < 0 \end{cases}$$

Relevance of the non-equilibrium effect in homogeneous shear turbulence

(Yoshizawa & Nisizima 1993, Phys. Fluids A**5**, 3302)

Homogeneous shear flow

$$\mathbf{U} = (U_x, U_y, U_z) = (Sy, 0, 0)$$

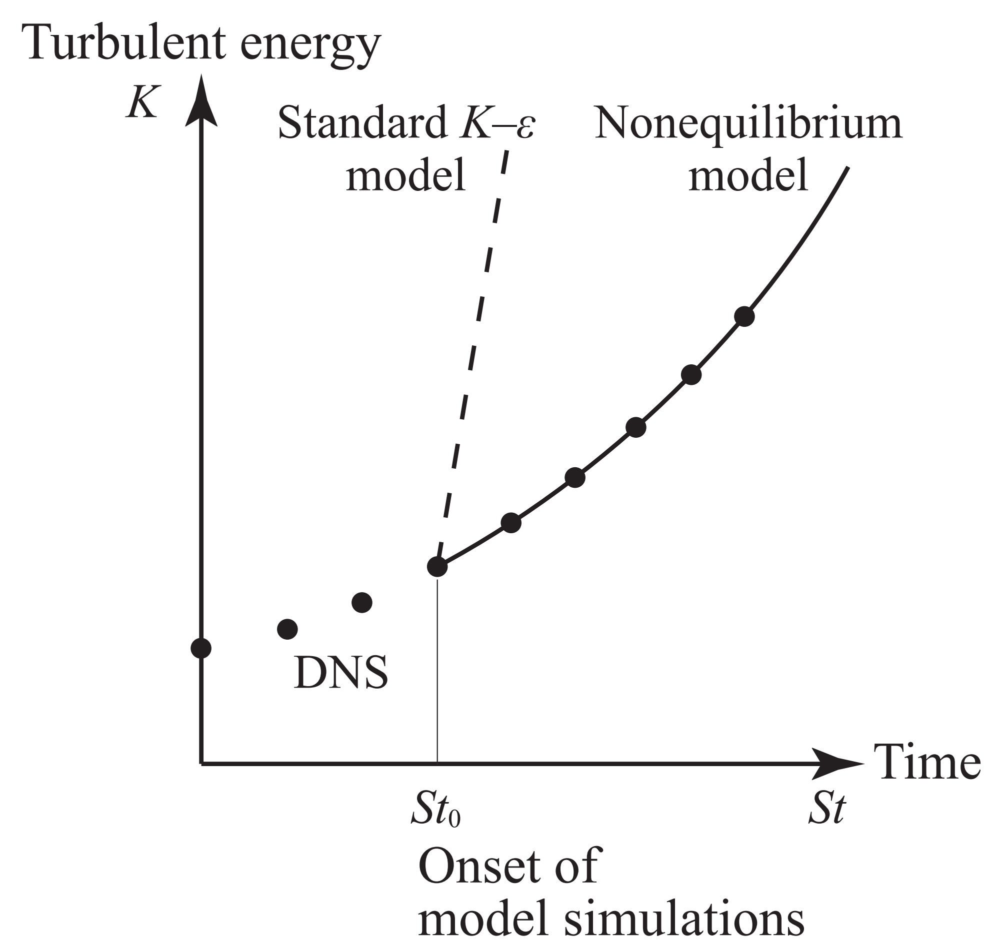
K–ε model

in homogeneous shear turbulence

$$\frac{\partial K}{\partial t} = P_K - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} = C_{\varepsilon 1} \frac{\varepsilon}{K} P_K - C_{\varepsilon 2} \frac{\varepsilon}{K} \varepsilon$$

with $P_K = -\left\langle u'_x u'_y \right\rangle \frac{dU_x}{dy} = +\nu_{\mathrm{T}} S^2$



Standard *K*–ε model with Equilibrium eddy viscosity

$$\nu_{\mathrm{T}} = \nu_{\mathrm{E}} = C_\nu \frac{K^2}{\varepsilon}$$

with $C_\nu = 0.09$

→

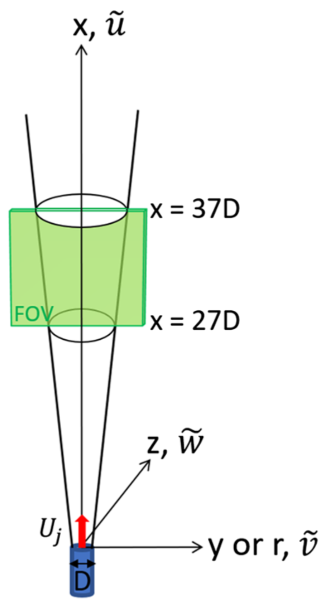
Non-equilibrium eddy viscosity

$$\nu_{\mathrm{T}} = \nu_{\mathrm{E}} \left(1 + C_{\mathrm{N}} \frac{1}{K} \frac{D}{Dt} \frac{K^2}{\varepsilon^2} \right)^{-1}$$

Non-equilibrium effects in experiment I

Round jet

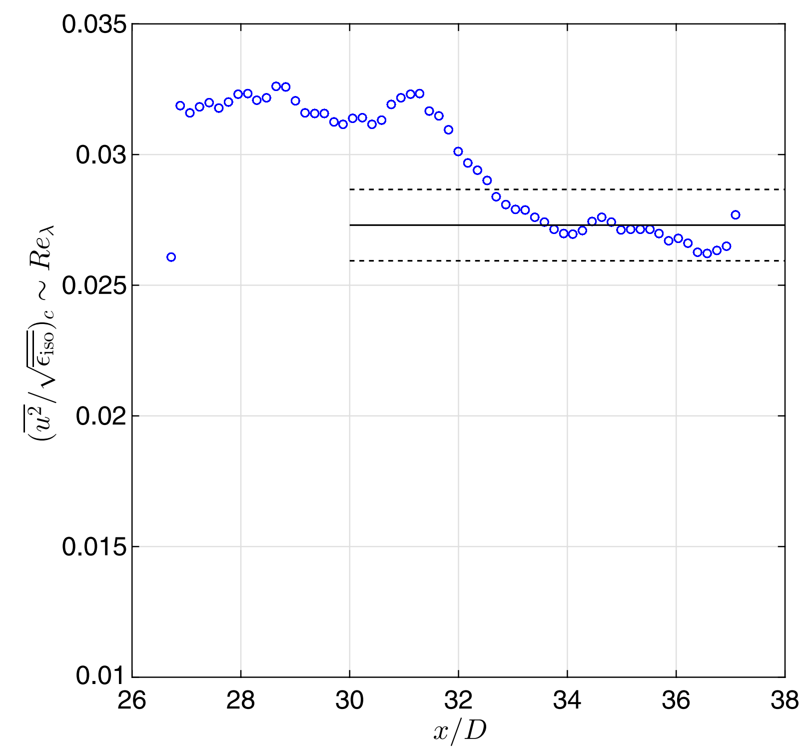
(Lai & Socolofsky 2019, Environ. Fluid Mech. **19**, 349)



Turbulent axial velocity fluctuation normalised by the square root of the dissipation rate at the jet centre

$$\frac{K^2}{\varepsilon} \propto \left[\frac{\langle (u'^x)^2 \rangle}{\sqrt{\varepsilon_{\rm iso}}}\right]^2$$

$$\frac{D}{Dt}\frac{K^2}{\varepsilon} < 0 \quad \text{(in turbulent round jets)}$$



x/D	$\sqrt{\langle (u'^x)^2\rangle}/U_{\rm c}$	$\sqrt{\langle (u'^y)^2\rangle}/U_{\rm c}$	$\sqrt{\langle (u'^z)^2\rangle}/U_{\rm c}$	$K/U_{\rm c}^2$	$\langle (u'^x)^2\rangle/\sqrt{\varepsilon_{\rm iso}}$
31	0.26	0.20	0.19	0.1437	3.25×10^{-2}
37	0.26	0.20	0.19	0.1437	2.65×10^{-2}
					(-20%)

$$\Lambda \simeq C_{\rm N}\frac{1}{K}(\mathbf{U}\cdot\boldsymbol{\nabla})\frac{K^2}{\varepsilon} \simeq C_{\rm N}\left(\frac{U_{\rm c}}{\sqrt{\langle (u'^x)^2\rangle}}\right)^2 \frac{1}{U_{\rm c}}\frac{\Delta\{[\langle (u'^x)^2\rangle/\sqrt{\langle \varepsilon_{\rm iso}\rangle}]^2\}}{D\Delta(x/D)} \simeq -1.7$$

$$\nu_{\rm NE} = \nu_{\rm E}\left(1 - C_{\rm N}\frac{1}{K}\frac{D}{Dt}\frac{K^2}{\varepsilon}\right) = \nu_{\rm E}\,(1-\Lambda) > \nu_{\rm E}$$

Eddy-viscosity-related constant

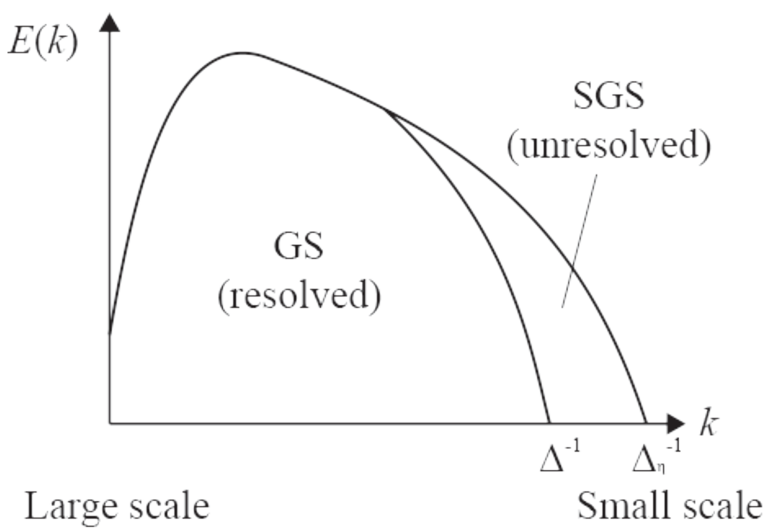
$$C_\mu = 0.09 \quad \rightarrow \quad 0.135$$

→ Transport enhancement

Double averaging approach

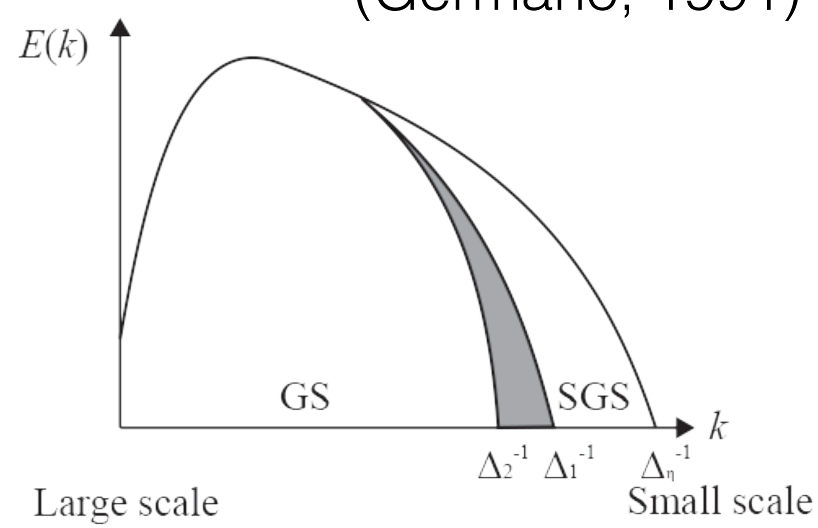
Double-averaging procedure

Subgrid-scale (SGS) modelling



Double-filtering approach

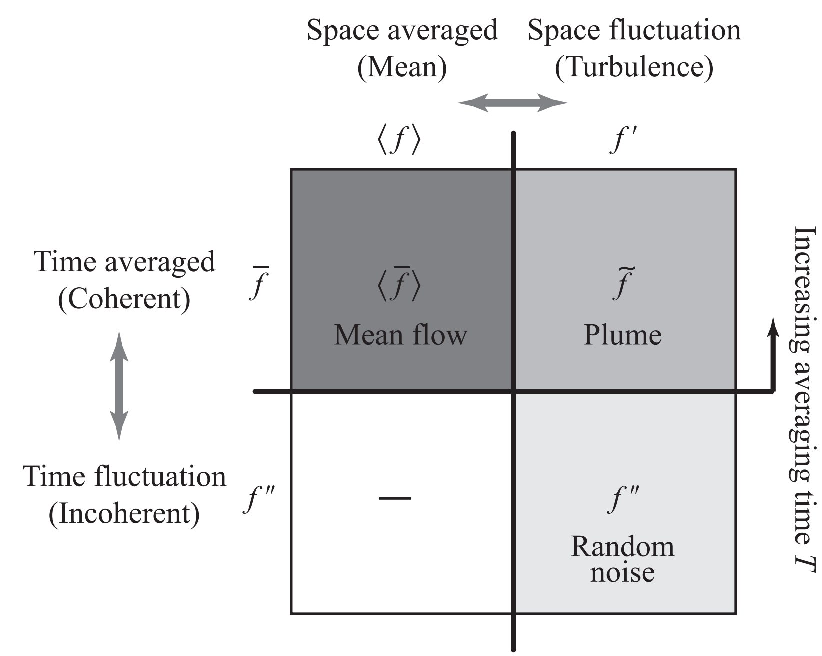
e.g., Dynamic Smagorinsky model (Germano, 1991)



Time–Space double filtering

Space-domain filtering

Time -domain filtering



Time–space double averaging

Filtering both in space and time domains

Time average

$$\overline{f}(\mathbf{x};t) = \int f(\mathbf{x};s)G(t-s)ds$$

Time filter

$$G(t-s) = \begin{cases} 1/T & (|t-s| \leq T/2) \\ 0 & (\text{otherwise}) \end{cases}$$

T: Time average window

$$\tau \ll T \ll \Xi.$$

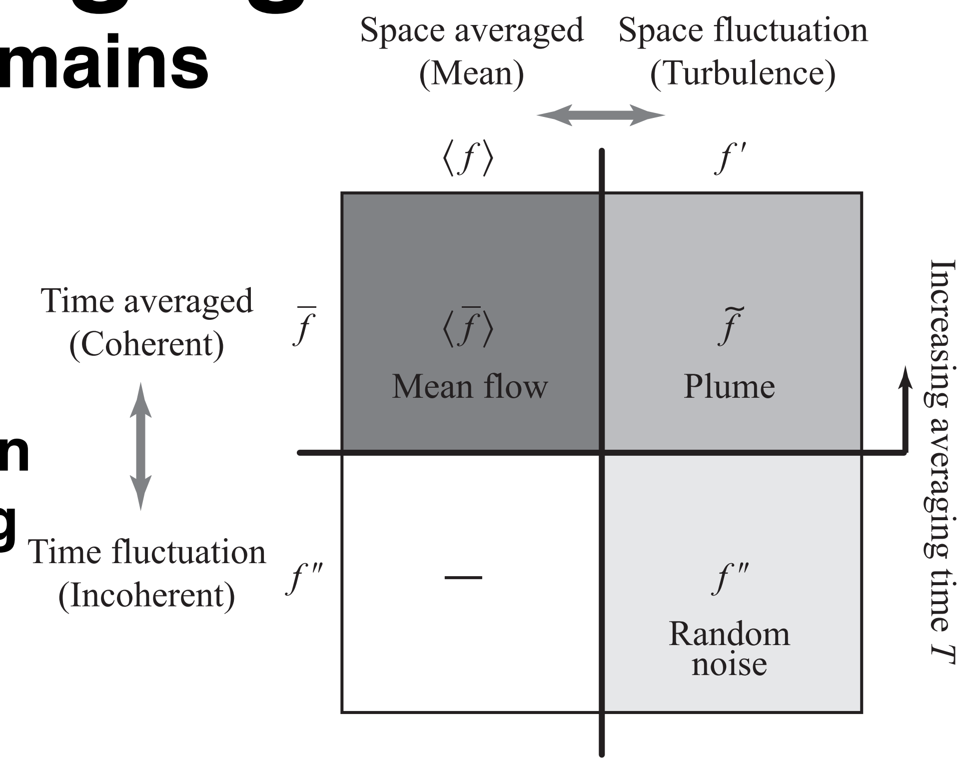
τ: Turn-over time Ξ: Mean-field evolution time

Space average

$$\langle f \rangle(z;t) = \frac{1}{\Delta x \Delta y}\int_S f(x,y,z;t)dxdy$$

Space-domain filtering

Time -domain filtering



$$f = \overbrace{\langle \overline{f} \rangle + \underbrace{\tilde{f}}_{\overline{f} - \langle \overline{f} \rangle}}^{\overline{f}} + \underbrace{f''}_{f - \overline{f}}$$

coherent fluctuation incoherent fluctuation

Coherent and incoherent fluctuations

$$f = \overbrace{\langle \overline{f} \rangle + \underbrace{\tilde{f}}_{\overline{f} - \langle \overline{f} \rangle}}^{\overline{f}} + \underbrace{f''}_{f - \overline{f}}$$

coherent fluctuation

incoherent fluctuation

Dispersion/Coherent fluctuation

$$\tilde{f} = \overline{f} - \langle \overline{f} \rangle$$

$$f' = \tilde{f} + f''$$

Rules of averaging

$$\langle \langle f \rangle \rangle = \langle f \rangle, \quad \overline{\overline{f}} = \overline{f} \qquad \langle f' \rangle = 0, \quad \langle \tilde{f} \rangle = 0, \quad \langle f'' \rangle = 0, \quad \overline{f''} = 0$$

$$\langle \overline{f} \rangle = \langle f \rangle, \quad \overline{\langle f \rangle} = \langle f \rangle$$

$$\langle \langle f \rangle \langle g \rangle \rangle = \langle f \rangle \langle g \rangle, \quad \overline{\overline{f}\,\overline{g}} = \overline{f}\,\overline{g}$$

$$\langle \overline{f}\,\overline{g} \rangle = \langle \overline{f} \rangle \langle \overline{g} \rangle = \langle f \rangle \langle g \rangle$$

$$\langle \tilde{f} g'' \rangle = 0, \quad \langle f'' \tilde{g} \rangle = 0,$$

$$\overline{\tilde{f} g} = \tilde{f}\overline{g}, \quad \overline{\tilde{f} g''} = 0$$

$$\overline{\tilde{f} g} = \overline{(\overline{f} - \langle \overline{f} \rangle) g} = \overline{\overline{f} g} - \overline{\langle \overline{f} \rangle g} = \overline{f}\,\overline{g} - \overline{\langle f \rangle}\,\overline{g}$$
$$= (\overline{f} - \langle \overline{f} \rangle)\overline{g} = \tilde{f}\overline{g}$$

Turbulent energy equations

Coherent/dispersion fluctuation

$$\left(\frac{\partial}{\partial t}+\langle u\rangle^{\ell}\frac{\partial}{\partial x^{\ell}}\right)\left\langle\frac{1}{2}(\tilde{u}^{a})^{2}\right\rangle$$

$$=-\langle\tilde{u}^{a}\tilde{u}^{\ell}\rangle\frac{\partial\langle u\rangle^{a}}{\partial x^{\ell}}-\frac{1}{\rho_{0}}\left\langle\tilde{p}\left(\frac{\partial\tilde{u}^{a}}{\partial x^{a}}\right)\right\rangle$$

$$-\nu\left\langle\left(\frac{\partial\tilde{u}^{a}}{\partial x^{\ell}}\right)^{2}\right\rangle+\frac{g}{\Theta_{0}}\left\langle\widetilde{\theta}\tilde{u}^{\ell}\right\rangle\delta^{\ell 3}$$

$$+\frac{\partial}{\partial x^{\ell}}\left\langle-\tilde{u}^{\ell}\frac{1}{2}(\tilde{u}^{a})^{2}+\tilde{p}\tilde{u}^{\ell}+\nu\frac{\partial}{\partial x^{\ell}}\frac{1}{2}(\tilde{u}^{a})^{2}\right\rangle$$

$$+\left\langle\widetilde{u''^{\ell}u''^{a}}\frac{\partial\tilde{u}^{a}}{\partial x^{\ell}}\right\rangle-\frac{\partial}{\partial x^{\ell}}\left\langle\widetilde{u''^{\ell}u''^{a}}\tilde{u}^{a}\right\rangle$$

Incoherent/random fluctuation

$$\left(\frac{\partial}{\partial t}+\langle u\rangle^{\ell}\frac{\partial}{\partial x^{\ell}}\right)\left\langle\frac{1}{2}(u''^{a})^{2}\right\rangle$$

$$=-\langle u''^{a}u''^{\ell}\rangle\frac{\partial\langle u\rangle^{a}}{\partial x^{\ell}}-\frac{1}{\rho_{0}}\left\langle p''\left(\frac{\partial u''^{a}}{\partial x^{a}}\right)\right\rangle$$

$$-\nu\left\langle\frac{\partial u''^{a}}{\partial x^{\ell}}\frac{\partial u''^{a}}{\partial x^{\ell}}\right\rangle+\frac{g}{\Theta_{0}}\langle\theta''u''^{\ell}\rangle\delta^{\ell 3}$$

$$+\frac{\partial}{\partial x^{\ell}}\left\langle-u''^{\ell}\frac{1}{2}(u''^{a})^{2}+p''u''^{\ell}+\nu\frac{\partial}{\partial x^{\ell}}\frac{1}{2}(u''^{a})^{2}\right\rangle$$

$$-\left\langle\widetilde{u''^{\ell}u''^{a}}\frac{\partial\tilde{u}^{a}}{\partial x^{\ell}}\right\rangle-\frac{\partial}{\partial x^{\ell}}\left\langle\widetilde{\frac{1}{2}(u''^{a})^{2}}\tilde{u}^{\ell}\right\rangle$$

$$\begin{aligned}\widetilde{u''^{\alpha}u''^{\beta}}&=\overline{u''^{\alpha}u''^{\beta}}-\langle\overline{u''^{\alpha}u''^{\beta}}\rangle\\&=\overline{u''^{\alpha}u''^{\beta}}-\langle u''^{\alpha}u''^{\beta}\rangle\end{aligned}$$

Dispersion part of the incoherent/random fluctuation correlation

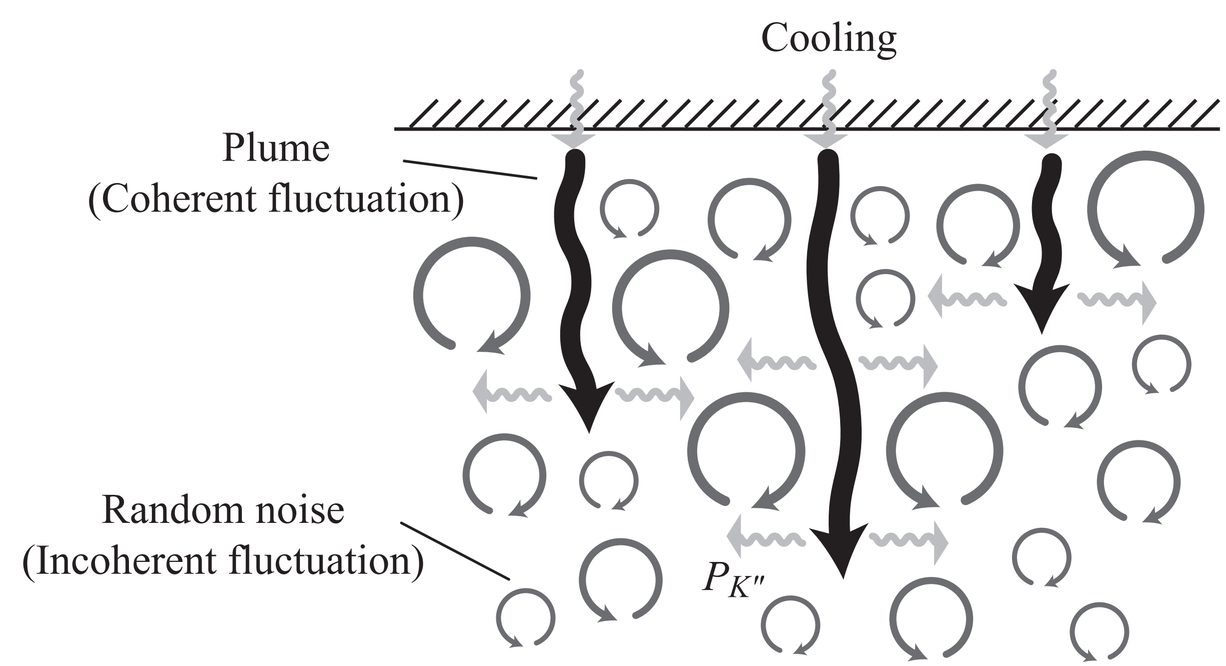
Modelling of transport with plumes

Scenario: Interaction between coherent and incoherent fluctuations

Source of incoherent fluctuation energy

Sink of coherent fluctuation energy

$$P_{K''} = -\left\langle \widetilde{u''^{\ell} u''^{m}} \frac{\partial \widetilde{u}^{m}}{\partial x^{\ell}} \right\rangle = -P_{\tilde{K}}$$



In the presence of **the coherent velocity shear**, energy transfer between the coherent and incoherent fluctuation components occurs mediated by **the dispersion part of the random fluctuation velocity correlation**.

Due to **the non-equilibrium effect**, timescale of coherent fluctuations is altered, leading to **an enhancement of energy transfer to random fluctuations** if $\Lambda < 0$.

$$\widetilde{u''^{\ell} u''^{m}} \simeq -\widetilde{\nu} \frac{\partial \widetilde{u}^{m}}{\partial x^{\ell}} + \text{N.E.}$$

$$P_{K''} = -\widetilde{u''^{\ell} u''^{m}} \frac{\partial \widetilde{u}^{m}}{\partial x^{\ell}} \simeq +\widetilde{\nu}(1 - \Lambda) \left(\frac{\partial \widetilde{u}^{m}}{\partial x^{\ell}} \right)^{2} > 0$$

$\Lambda < 0$ Enhancement of energy transfer to random fluctuation

Turbulence modeling in the time–space double averaging

Turbulent internal-energy flux

$$\left\langle \overline{e'\mathbf{u}'} \right\rangle = \left\langle \overline{\tilde{e}\tilde{\mathbf{u}}} \right\rangle + \left\langle \overline{e''\mathbf{u}''} \right\rangle$$

$$= -\left\langle \tilde{\kappa}_{\mathrm{E}} \left(1 - \tilde{C}\tilde{\tau} \frac{1}{\overline{\tilde{\mathbf{u}}^2}} \frac{\overline{D}\overline{\tilde{\mathbf{u}}^2}}{\overline{D}t} \right) \nabla E \right\rangle - \left\langle \kappa''_{\mathrm{E}} \left(1 - C''\tau'' \frac{1}{\overline{\mathbf{u}''^2}} \frac{\overline{D}\overline{\mathbf{u}''^2}}{\overline{D}t} \right) \nabla E \right\rangle$$

$$\langle e'\mathbf{u}' \rangle = -\kappa_{\mathrm{NE}} \nabla E$$

$$\left\langle \overline{\mathbf{u}'^2} \right\rangle = \left\langle \overline{\tilde{\mathbf{u}}^2} \right\rangle + \left\langle \overline{\mathbf{u}''^2} \right\rangle$$

$$\left\langle \overline{\tilde{\mathbf{u}}^2} \right\rangle \simeq \left\langle \overline{\mathbf{u}''^2} \right\rangle \simeq \left\langle \overline{\mathbf{u}'^2} \right\rangle / 2$$

Coherent timescale Incoherent timescale

$$\tilde{\tau} \gg \tau''$$

$$\tilde{\varepsilon} = \frac{\langle \tilde{\mathbf{u}}^2 \rangle}{\tilde{\tau}} \ll \frac{\langle \mathbf{u}''^2 \rangle}{\tau''} = \varepsilon''$$

Turbulent diffusivity with non-equilibrium effect

$$\kappa_{\mathrm{NE}} = \begin{cases} \kappa_{\mathrm{E}} \left[1 - C_{\tilde{\tau}} \dfrac{\tilde{\tau}}{\langle \mathbf{u}'^2 \rangle} \tilde{\Lambda}_D \right] & \text{for} \quad \tilde{\Lambda}_D < 0 \\ \kappa_{\mathrm{E}} \left[1 + C_{\tilde{\tau}} \dfrac{\tilde{\tau}}{\langle \mathbf{u}'^2 \rangle} \tilde{\Lambda}_D \right]^{-1} & \text{for} \quad \tilde{\Lambda}_D > 0 \end{cases}$$

Non-equilibrium property along the plume motion

$$\tilde{\Lambda}_D = \left\langle (\tilde{\mathbf{u}} \cdot \nabla) \overline{\mathbf{u}'^2} \right\rangle$$

Application to stellar convection

Set-up of the convection simulation

Polytropic gas $p = \rho^{1+\frac{1}{m}}$

Eq. of State $p = (\gamma - 1)\rho e$ $\quad e = \dfrac{p}{(\gamma-1)\rho} = \dfrac{1}{\gamma - 1}\rho^{1/m}$ $\quad p = \rho^{\frac{m+1}{m}} = (\gamma-1)^{m+1} e^{m+1}$

Hydrostatic balance

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = 0 \quad \longrightarrow \quad \rho = \rho_\mathrm{s}\left(\frac{e}{e_\mathrm{s}}\right)^m \qquad p = p_\mathrm{s}\left(\frac{e}{e_\mathrm{s}}\right)^{m+1}$$

$$e = e_\mathrm{s} + \frac{g}{(\gamma-1)(m+1)}(z_\mathrm{s} - z)$$

Determining the spatial distribution of the density and pressure

Convection instability condition

$\nabla - \nabla_\mathrm{ad} > 0$ where $\nabla = \dfrac{\partial \ln \theta}{\partial \ln p} = \dfrac{p}{\theta}\dfrac{\partial \theta}{\partial p}$ $\quad \nabla_\mathrm{ad} = \left(\dfrac{\partial \ln \theta}{\partial \ln p}\right)_\mathrm{ad} = \left(\dfrac{p}{\theta}\dfrac{\partial \theta}{\partial p}\right)_\mathrm{ad}$

For a polytropic gas $\dfrac{\partial p}{\partial \theta} = (m+1)\dfrac{p}{\theta}$ $\quad \nabla = \dfrac{\partial \ln \theta}{\partial \ln p} = \dfrac{p}{\theta}\dfrac{\partial \theta}{\partial p} = \dfrac{1}{m+1}$

In the adiabatic case

$$e = \frac{1}{\gamma-1}\rho^{\gamma-1} = \frac{1}{\gamma-1}p^{1-\frac{1}{\gamma}} \qquad \nabla_\mathrm{ad} = \frac{\partial \ln e}{\partial \ln p} = \frac{\partial \ln \theta}{\partial \ln p} = 1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma}$$

Instability condition

$$m < \frac{\gamma}{\gamma - 1} - 1 \quad \xrightarrow{\gamma = 5/3}$$

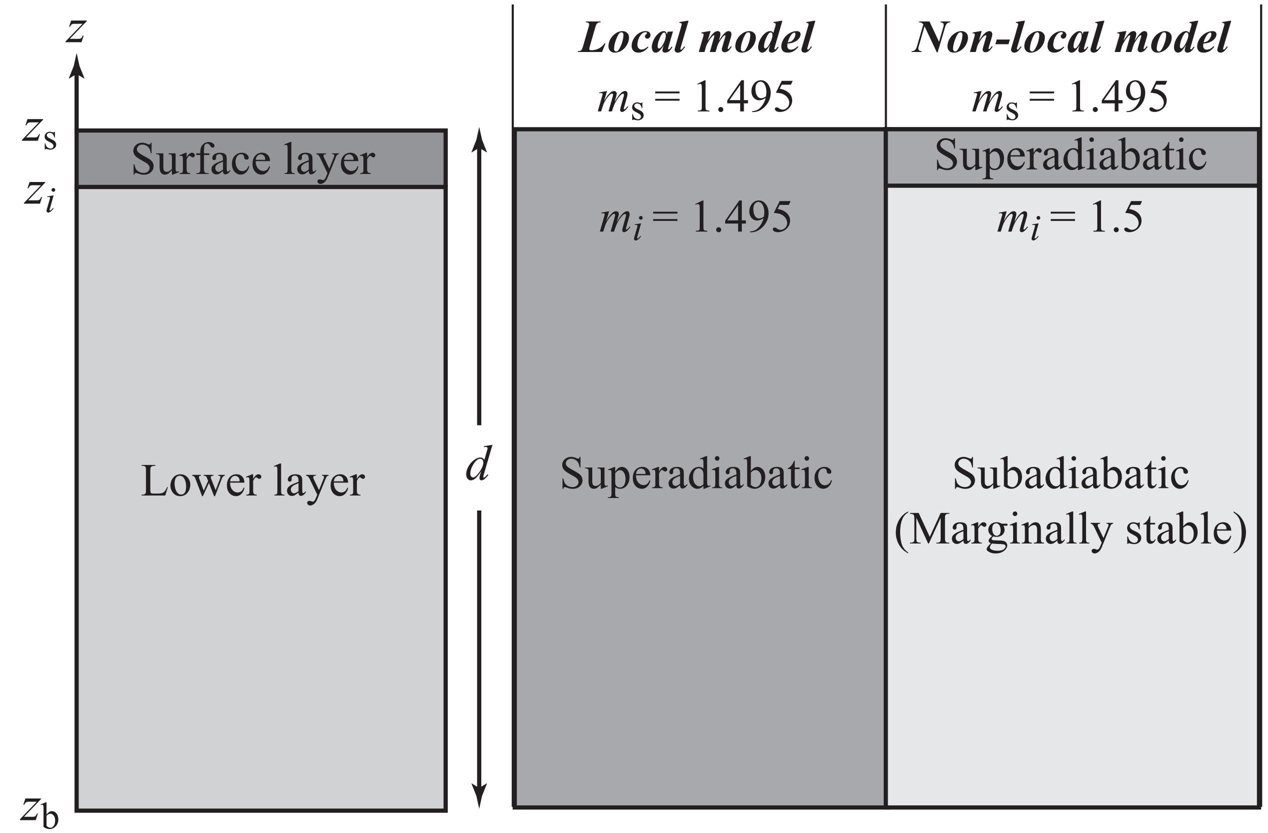
$m < 1.5$ Unstable

$m = 1.5$ Marginally stable

Two-layer polytropic gas convection

(Cossette & Rast 2016, ApJL, 829, L17)

$$\begin{cases} e(z_i \le z \le z_\mathrm{s}) = e_\mathrm{s} + \dfrac{g(z_\mathrm{s} - z)}{(\gamma - 1)(m_\mathrm{s} + 1)} \\ e(z_\mathrm{b} \le z \le z_i) = e_i + \dfrac{g(z_i - z)}{(\gamma - 1)(m_i + 1)} \end{cases}$$



Control parameter: e_s

specific internal energy at the surface (z_S)

throughout the convection zone

near surface region

Local model: $m_\mathrm{s} = m_i = 1.495$

Non-local model: $m_\mathrm{s} = 1.495, \;\; m_i = 1.5 \qquad z_i/d = 0.95$

$$\nu = \eta = 1 \times 10^{-4}, \;\; \kappa = 1 \times 10^{-4} \;\; \text{(conductivity)}$$

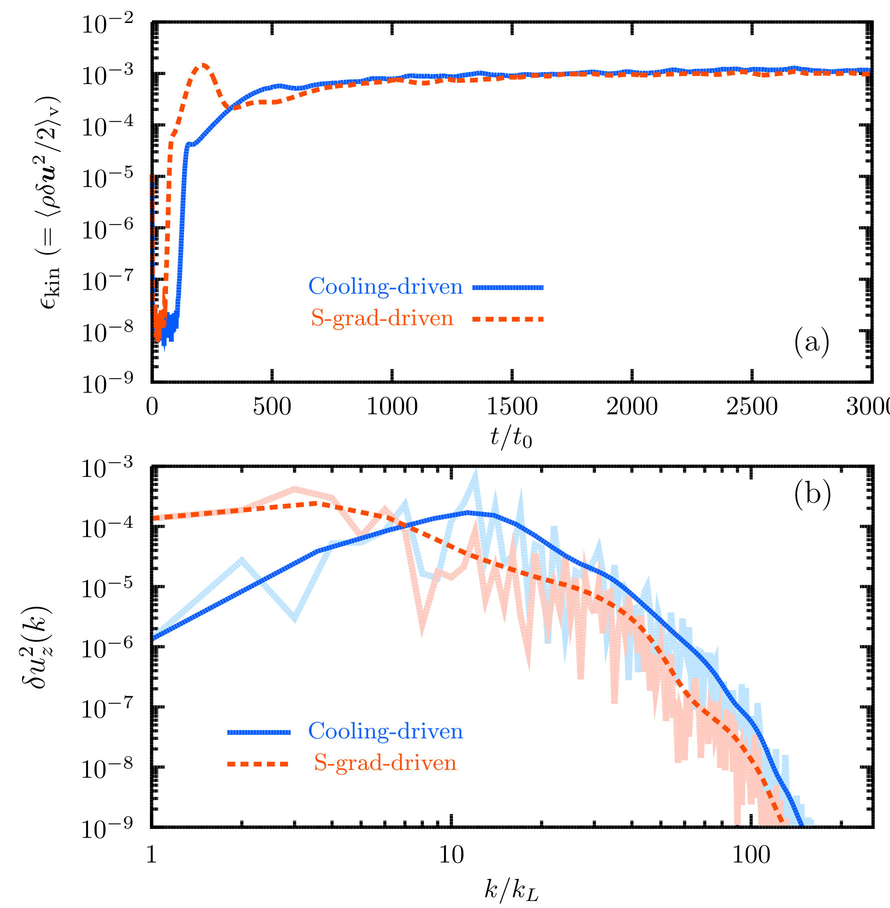
Non-dimensional parameters $\quad Pr = \dfrac{\nu}{\chi}, \;\; Pm = \dfrac{\nu}{\eta}, \;\; Ra = \dfrac{g d^4 \delta}{\chi \nu H_\rho} \quad$ with $\quad \chi = \dfrac{\kappa}{\langle \rho \rangle}$

$$Pr \simeq 1, \;\; Pm = 1, \;\; Ra \simeq 4.2 \times 10^6 \qquad \rho_\mathrm{bottom}/\rho_\mathrm{top} = 100$$

Results

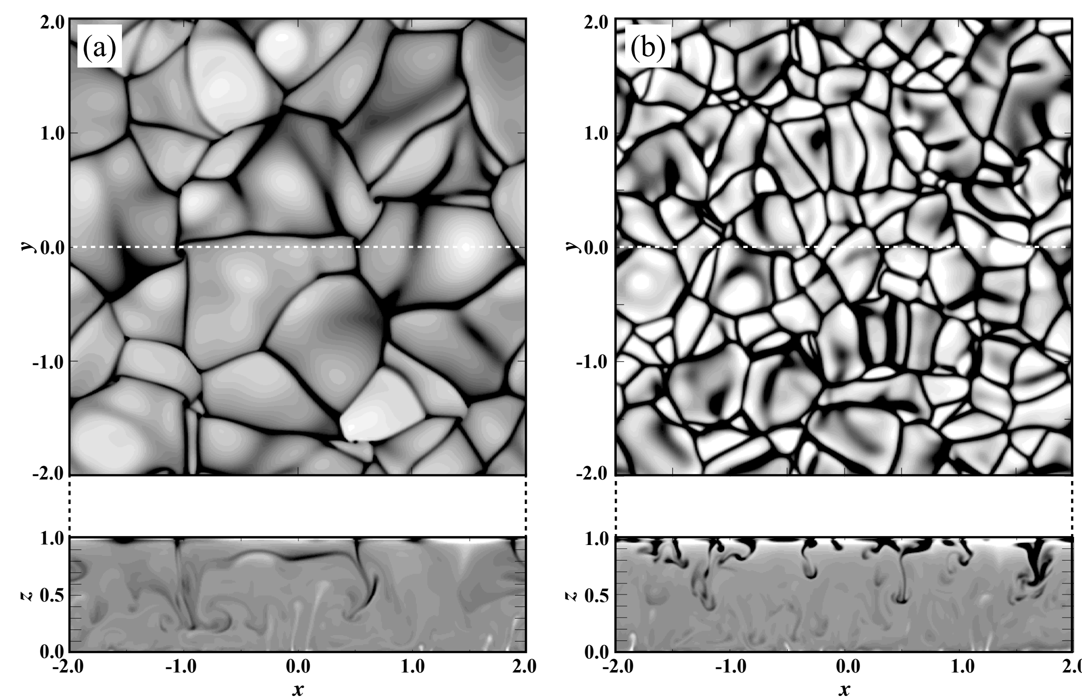
Time evolution of kinetic energy

Energy spectra



Entropy distributions

Locally driven case **Non-locally driven case**



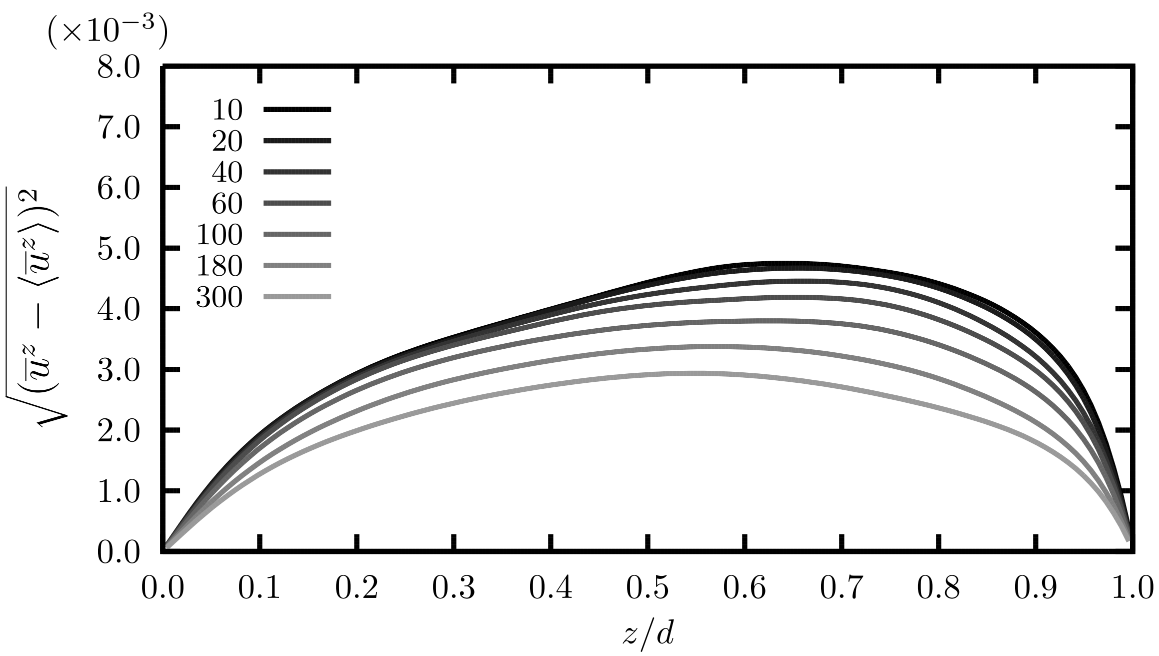
Horizontal cross-sections of the entropy fluctuation at the top surface

Vertical cross-sections of the entropy fluctuation from the horizontal mean

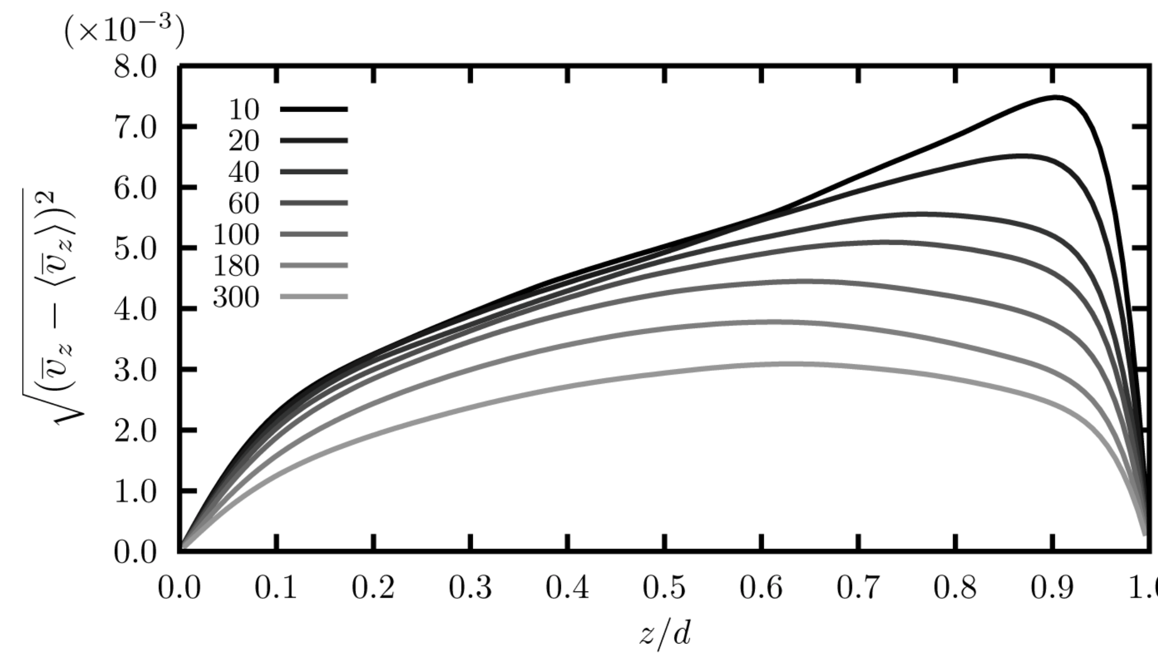
Coherent fluctuations

$$\sqrt{(\tilde{u}^z)^2} = \sqrt{(\overline{u}^z - \langle \overline{u}^z \rangle)^2}$$

Local (superadiabatic driven)



Non-local (surface cooling driven)



Lifetime of plume and averaging time

Non-dimensionalised horizontally averaged velocity is about $v^z = \sqrt{\langle (u^z)^2 \rangle} \sim 0.01$

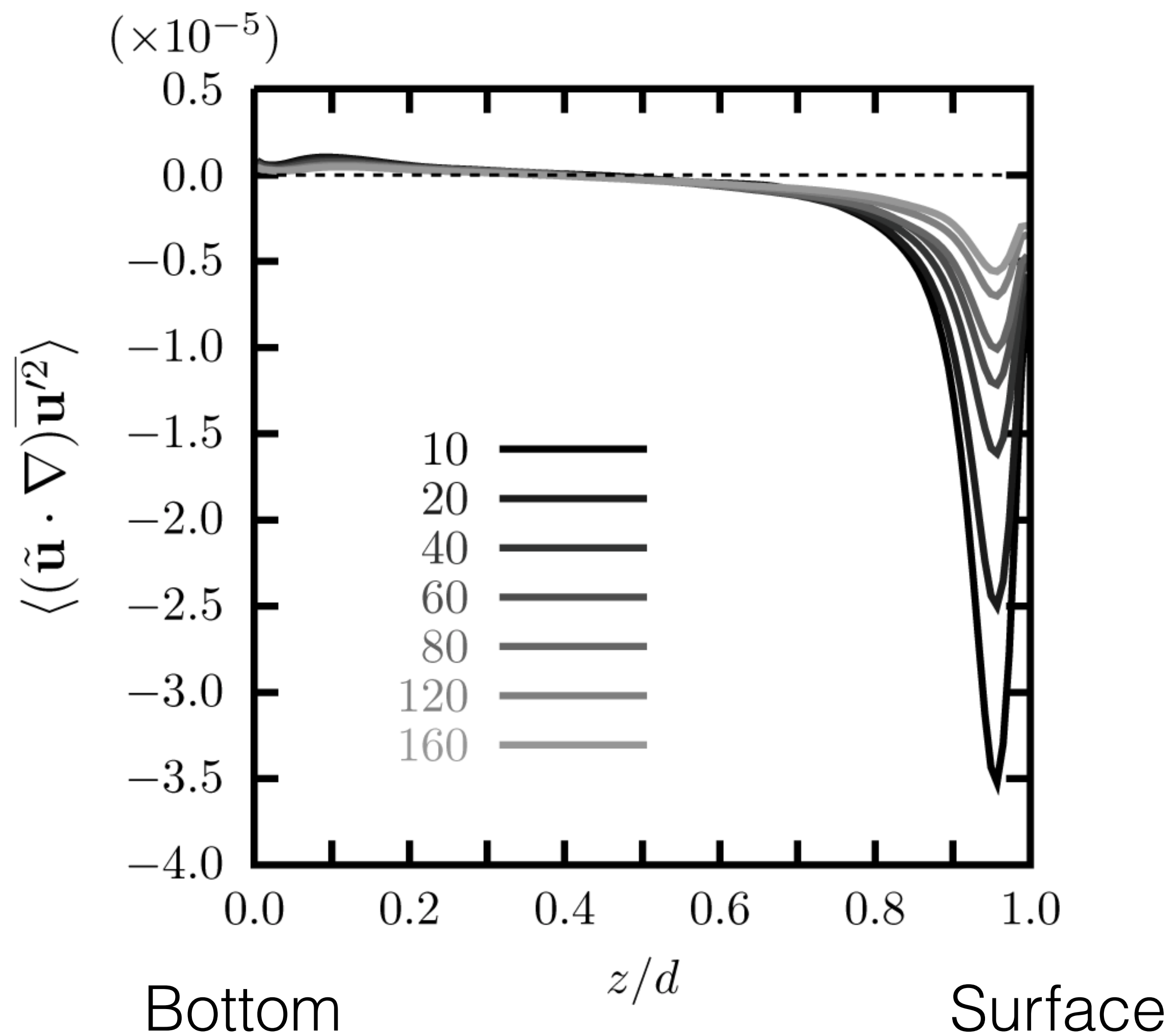
Non-dimensionalised full depth of the convection zone $d = 1$

Lifetime of the plume may be estimated (from crossing time) as $\dfrac{d/2}{v^z} \sim \dfrac{0.5}{0.01} \sim 50$

→ Averaging time $T \lesssim 25$

Non-equilibrium property along the plume motion

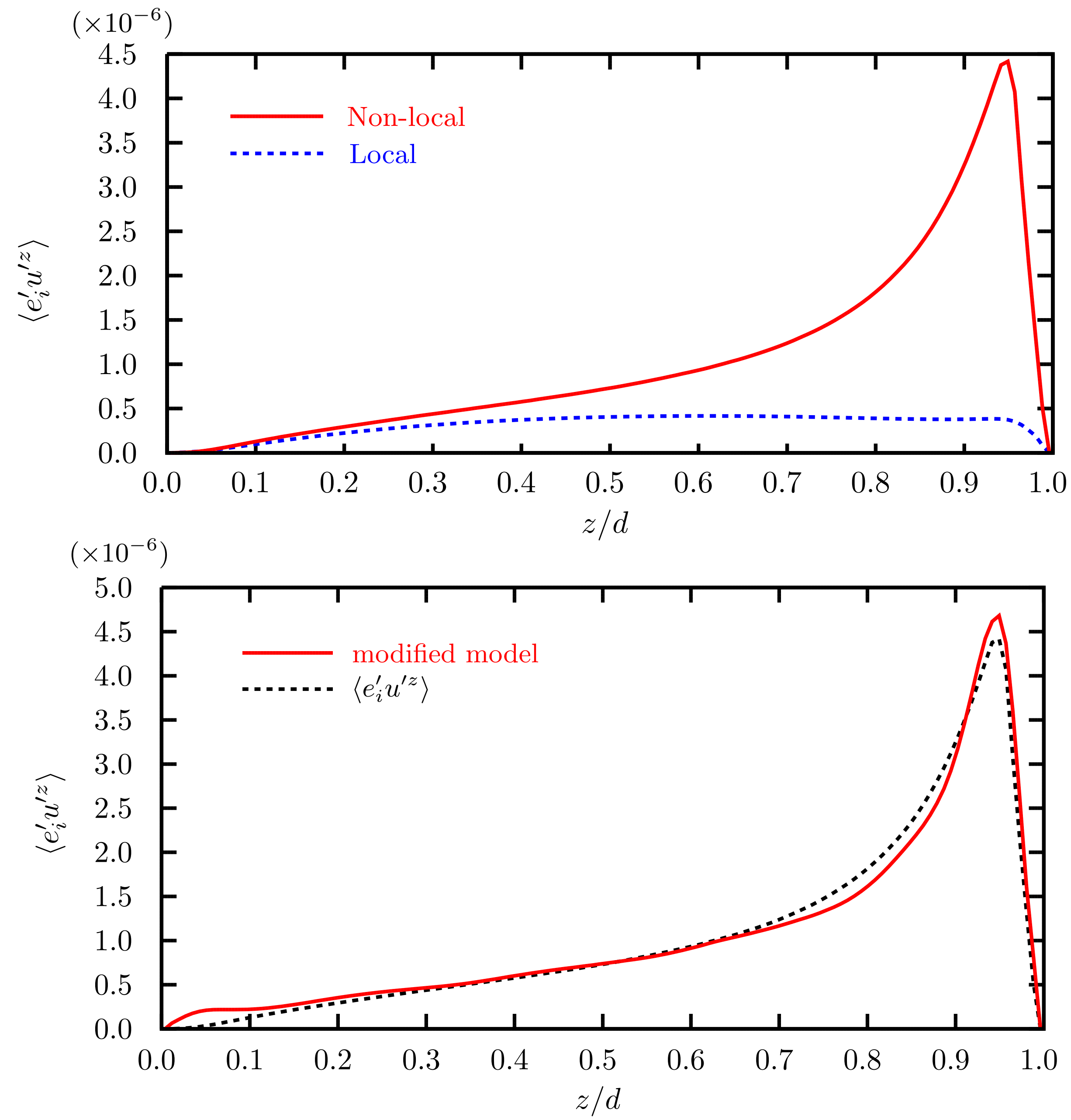
$$\tilde{\Lambda}_D = \left\langle (\tilde{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}'^2} \right\rangle$$



Turbulent energy flux $\langle e' u'^z \rangle$

by DNS

by Non-equilibrium model

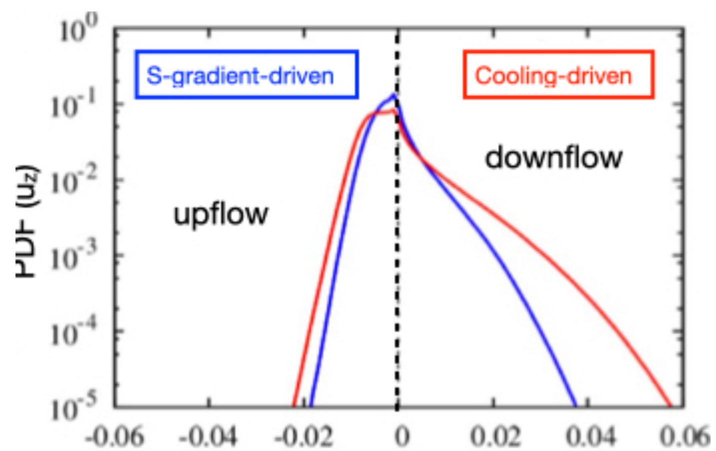


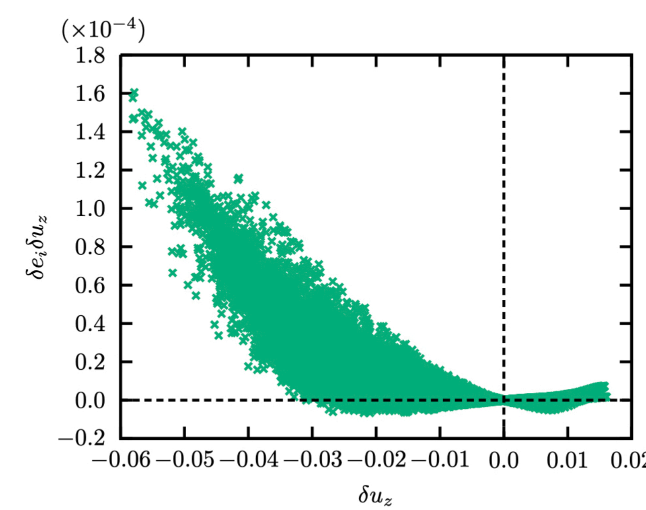
A similar result is obtained for the turbulent mass flux $\langle \rho' u'^z \rangle$

The **non-equilibrium effect** in the time–space **double averaging** framework is a promising model approach to the **stellar convection with plume**.

Summary of non-equilibrium effects on convective plumes

- Transport due to plume motion is incorporated into turbulence model through the **non-equilibrium effect through the time variation of fluctuations along the plume motion**
- Turbulence modelling in the **time–space double averaging framework**
- Turbulent convective transport of **surface cooling diving plumes (non-local transport) is properly described by the non-equilibrium model**





(Masada, Private communication)

Wish list

Physics of plume formation

Statistical properties (probability distributions)

Dynamical properties (length, aspect ratio, shape)

What determines the non-equilibrium properties of turbulence

Entropy production rate etc.

Non-equilibrium effect in dynamos

Cross-interaction responses

Yokoi, N. "Unappreciated cross-helicity effects in plasma physics: Anti-diffusion effects in dynamo and momentum transport," **Rev. Mod. Plasma Phys. 7, 33-1-98 (2023)**
https://doi.org/10.1007/s41614-023-00133-4

Mizerski, K., Yokoi, N. & Brandenburg, A. "Cross-helicity effect on α-type dynamo in non-equilibrium turbulence," **J. Plasma Phys. 89, 905890412 (2023)**
https://doi.org/10.1017/S0022377823000545

Global flow generation by cross helicity

Reynolds and turbulent Maxwell stress

eddy viscosity   inhomogeneous helicity

Theoretical Result $\langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}' \rangle_{\mathrm{D}} = -\nu_{\mathrm{K}} \boldsymbol{\mathcal{S}} + \nu_{\mathrm{M}} \boldsymbol{\mathcal{M}} + \eta_H \boldsymbol{\Omega}_* \nabla H + \cdots$

cross helicity

D: deviatoric part

$\boldsymbol{\mathcal{S}}$: mean velocity strain $\boldsymbol{\mathcal{S}} = \nabla \mathbf{U} + (\nabla \mathbf{U})^{\dagger}$

$\boldsymbol{\mathcal{M}}$: mean magnetic-field strain $\boldsymbol{\mathcal{M}} = \nabla \mathbf{B} + (\nabla \mathbf{B})^{\dagger}$

$\boldsymbol{\Omega}_*$: absolute mean vorticity (mean vorticity + rotation)

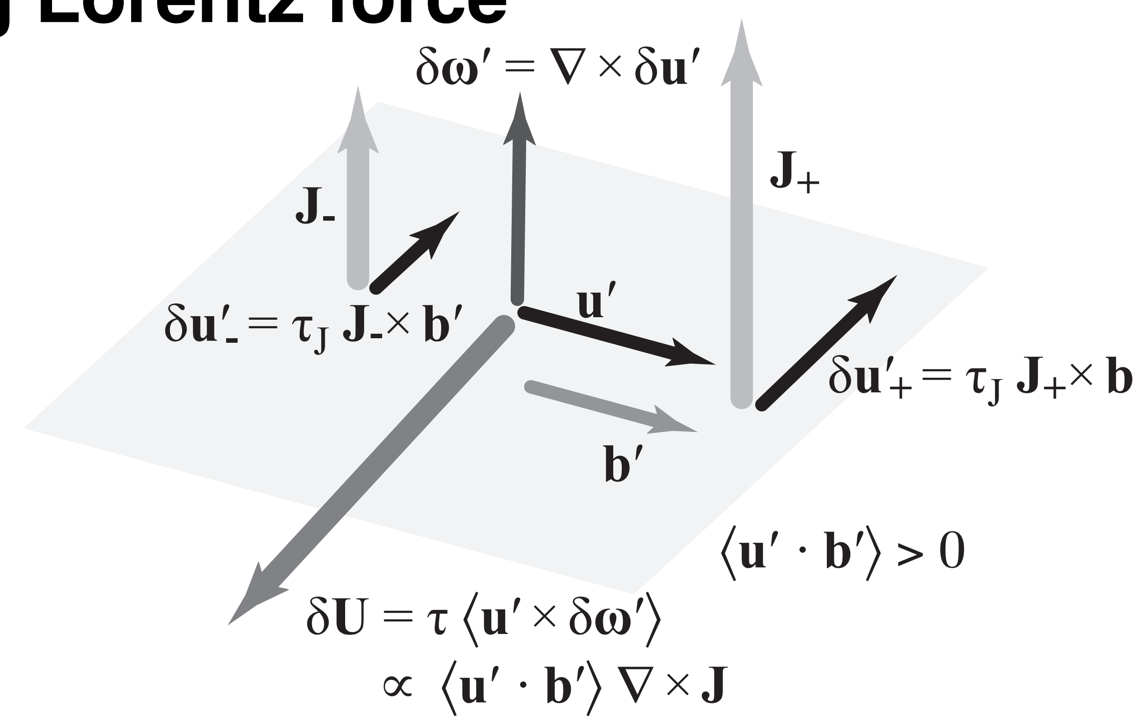
cf. $\langle \mathbf{u}' \times \mathbf{b}' \rangle = -\eta_{\mathrm{T}} \nabla \times \mathbf{B} + \gamma \nabla \times \mathbf{U} + \alpha \mathbf{B} + \cdots$

Turbulent cross helicity coupled with **mean magnetic-field strain** may contribute to **transport suppression** and/or **global flow generation** against the eddy-viscosity effect

Physical interpretation of large-scale flow generation by cross helicity

Velocity fluctuation
induced by **fluctuating Lorentz force**

$$\delta \mathbf{u}' = \tau_J \mathbf{J} \times \mathbf{b}'$$



Associated vorticity

$$\begin{aligned} \delta\boldsymbol{\omega}' &= \nabla \times \delta\mathbf{u}' \\ &= \tau_J \nabla \times (\mathbf{J} \times \mathbf{b}') \\ &\simeq \tau_J (\mathbf{b}' \cdot \nabla)\mathbf{J} \end{aligned}$$

Non-trivial mean electric-current distribution
(Inhomogeneous **J**) is required

$\delta\mathbf{U} = \tau \langle \mathbf{u}' \times \delta\boldsymbol{\omega}' \rangle \propto \langle \mathbf{u}' \cdot \mathbf{b}' \rangle \nabla \times \mathbf{J}$ in the direction of $\nabla \times \mathbf{J}$

Theoretical formulation (e.g., incompressible MHD)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = (\mathbf{b}\cdot\nabla)\mathbf{b} - \nabla p_{\mathrm{M}} - 2\boldsymbol{\omega}_{\mathrm{F}}\times\mathbf{u} + \nu\nabla^2\mathbf{u}$$

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} = (\mathbf{b}\cdot\nabla)\mathbf{u} + \eta\nabla^2\mathbf{b}$$

$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{b} = 0$$

Multiple-scale analysis $\boldsymbol{\xi}(=\mathbf{x}),\ \mathbf{X}(=\delta\mathbf{x}),\ \tau(=t),\ T(=\delta t)$

$$\nabla_{\mathbf{x}} = \nabla_{\boldsymbol{\xi}} + \delta\nabla_{\mathbf{X}};\quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \delta\frac{\partial}{\partial T}$$

$$f(\mathbf{x};t) = F(\mathbf{X};T) + f'(\boldsymbol{\xi},\mathbf{X};\tau,T)$$

Two-scale equations in configuration space

$$\frac{\partial u'^i}{\partial \tau} + U^j\frac{\partial u'^i}{\partial \xi^j} + \frac{\partial}{\partial \xi^j}(u'^j u'^i - b'^j b'^i) + \frac{\partial p'_{\mathrm{M}}}{\partial \xi^i} - \nu\frac{\partial^2 u'^i}{\partial \xi^j \partial \xi^j} - B^j\frac{\partial b'^i}{\partial \xi^j}$$
$$= \delta\left[b'^j\frac{\partial B^i}{\partial X^j} - u'^j\left(\frac{\partial U^i}{\partial X^j} + \epsilon^{jik}\Omega_0^k\right) + B^j\frac{\partial b'^i}{\partial X^j} - \frac{\overline{D}u'^i}{DT} - \frac{\partial}{\partial X^j}\left(u'^j u'^i - b'^j b'^i - \langle u'^j u'^i - b'^j b'^i\rangle\right) - \frac{\partial p'_{\mathrm{M}}}{\partial X^i}\right]$$

$$\frac{\partial u'^i}{\partial \xi^i} + \delta\frac{\partial u'^i}{\partial X^i} = 0,$$

$$\frac{\partial b'^i}{\partial \tau} + U^j\frac{\partial b'^i}{\partial \xi^j} + \frac{\partial}{\partial \xi^j}(u'^j b'^i - b'^j u'^i) - \eta\frac{\partial^2 b'^i}{\partial \xi^j \partial \xi^j} - B^j\frac{\partial u'^i}{\partial \xi^j}$$
$$= \delta\left[u'^j\frac{\partial B^i}{\partial X^j} - b'^j\left(\frac{\partial U^i}{\partial X^j} + \epsilon^{jik}\Omega_0^k\right) + B^j\frac{\partial u'^i}{\partial X^j} - \frac{\overline{D}b'^i}{DT} - \frac{\partial}{\partial X^j}\left(u'^j b'^i - b'^j u'^i - \langle u'^j b'^i - b'^j u'^i\rangle\right)\right]$$

$$\frac{\partial b'^i}{\partial \xi^i} + \delta\frac{\partial b'^i}{\partial X^i} = 0$$

inhomogeneities,
anisotropies,
non-equilibrium properties

Scale parameter expansion

$$f^i(\mathbf{k};\tau) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \delta^n f^i_{nm}(\mathbf{k};\tau) - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \delta^{n+1} i \frac{k^i}{k^2} \frac{\partial}{\partial X^j_{\mathrm{I}}} f^j_{nm}(\mathbf{k};\tau)$$

Basic-field (lowest-order field) equations

$$\frac{\partial u^i_{00}(\mathbf{k};\tau)}{\partial \tau} + \nu k^2 u^i_{00}(\mathbf{k};\tau) - iM^{ijk}(\mathbf{k}) \iint d\mathbf{p}d\mathbf{q}\ \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \left[u^j_{00}(\mathbf{p};\tau)u^k_{00}(\mathbf{q};\tau) - b^j_{00}(\mathbf{p};\tau)b^k_{00}(\mathbf{q};\tau)\right] = 0$$

$$\frac{\partial b^i_{00}(\mathbf{k};\tau)}{\partial \tau} + \eta k^2 b^i_{00}(\mathbf{k};\tau) - iN^{ijk}(\mathbf{k}) \iint d\mathbf{p}d\mathbf{q}\ \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \left[u^j_{00}(\mathbf{p};\tau)b^k_{00}(\mathbf{q};\tau) - b^j_{00}(\mathbf{p};\tau)u^k_{00}(\mathbf{q};\tau)\right] = 0$$

Projection operators

$$M^{ijk}(\mathbf{k}) = \frac{1}{2}\left[k^j D^{ik}(\mathbf{k}) + k^k D^{ij}(\mathbf{k})\right], \quad D^{ij}(\mathbf{k}) = \delta^{ij} - \frac{k^i k^j}{k^2}$$

$$N^{ijk}(\mathbf{k}) = k^j \delta^{ik} - k^k \delta^{ij}$$

$f_{01}(\mathbf{k};\tau)$ equation

$$\begin{pmatrix} \frac{\partial}{\partial\tau}+\nu k^2 & 0 \\ 0 & \frac{\partial}{\partial\tau}+\eta k^2 \end{pmatrix} \begin{pmatrix} u^i_{01}(\mathbf{k};\tau) \\ b^i_{01}(\mathbf{k};\tau) \end{pmatrix}$$

$$+i \begin{pmatrix} -2M^{ikm}(\mathbf{k})\int_\Delta u^k_{00}(\mathbf{p};\tau) & 2M^{ikm}(\mathbf{k})\int_\Delta b^k_{00}(\mathbf{p};\tau) \\ N^{ikm}(\mathbf{k})\int_\Delta b^k_{00}(\mathbf{p};\tau) & -N^{ikm}(\mathbf{k})\int_\Delta u^k_{00}(\mathbf{p};\tau) \end{pmatrix} \begin{pmatrix} u^m_{01}(\mathbf{q};\tau) \\ b^m_{01}(\mathbf{q};\tau) \end{pmatrix}$$

$$= -ik^jB^j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u^i_{00}(\mathbf{k}) \\ b^i_{00}(\mathbf{k}) \end{pmatrix}$$

with $\int_\Delta = \iint d\mathbf{p}d\mathbf{q}\ \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})$

$f_{10}(\mathbf{k};\tau)$ equation

$$\begin{pmatrix} \frac{\partial}{\partial\tau}+\nu k^2 & 0 \\ 0 & \frac{\partial}{\partial\tau}+\eta k^2 \end{pmatrix} \begin{pmatrix} u^i_{10}(\mathbf{k};\tau) \\ b^i_{10}(\mathbf{k};\tau) \end{pmatrix}$$

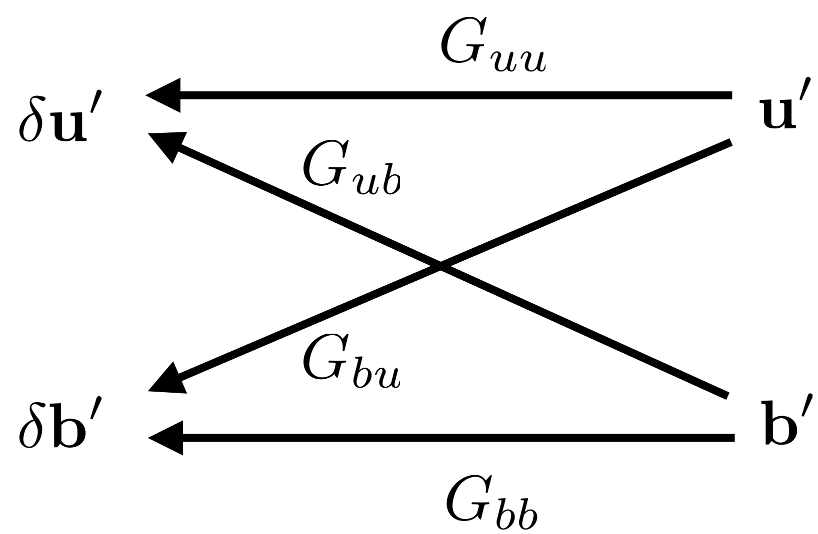
$$+i \begin{pmatrix} -2M^{ikm}(\mathbf{k})\int_\Delta u^k_{00}(\mathbf{p};\tau) & 2M^{ikm}(\mathbf{k})\int_\Delta b^k_{00}(\mathbf{p};\tau) \\ N^{ikm}(\mathbf{k})\int_\Delta b^k_{00}(\mathbf{p};\tau) & -N^{ikm}(\mathbf{k})\int_\Delta u^k_{00}(\mathbf{p};\tau) \end{pmatrix} \begin{pmatrix} u^m_{10}(\mathbf{q};\tau) \\ b^m_{10}(\mathbf{q};\tau) \end{pmatrix}$$

$$= B^k\frac{\partial}{\partial X^k_{\mathrm{I}}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u^i_{00}(\mathbf{k}) \\ b^i_{00}(\mathbf{k}) \end{pmatrix} - D^{jk}(\mathbf{k})\frac{\overline{D}}{DT_{\mathrm{I}}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^k_{00}(\mathbf{k}) \\ b^k_{00}(\mathbf{k}) \end{pmatrix}$$

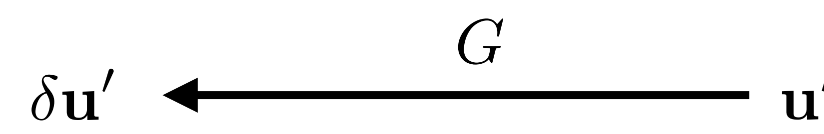
$$+ \begin{pmatrix} -D^{jk}(\mathbf{k})\left(\frac{\partial U^k}{\partial X^m}+\epsilon^{mkn}\Omega^n_0\right) & D^{jk}(\mathbf{k})\frac{\partial B^k}{\partial X^m} \\ -D^{jk}(\mathbf{k})\frac{\partial B^k}{\partial X^m} & D^{jk}(\mathbf{k})\left(\frac{\partial U^k}{\partial X^m}+\epsilon^{mkn}\Omega^n_0\right) \end{pmatrix} \begin{pmatrix} u^m_{00}(\mathbf{k}) \\ b^m_{00}(\mathbf{k}) \end{pmatrix}$$

Response functions in MHD

Cross-interaction responses in incompressible MHD



cf. Hydrodynamic case



Green's function equations

$$\begin{pmatrix} \dfrac{\partial}{\partial\tau}+\nu k^2 & 0 \\ 0 & \dfrac{\partial}{\partial\tau}+\eta k^2 \end{pmatrix} \begin{pmatrix} G^{ij}_{uu} & G^{ij}_{bu} \\ G^{ij}_{ub} & G^{ij}_{bb} \end{pmatrix}$$

$$+i \begin{pmatrix} -2M^{ikm}\displaystyle\int_\Delta u^k_{00} & 2M^{ikm}\displaystyle\int_\Delta b^k_{00} \\ N^{ikm}\displaystyle\int_\Delta b^k_{00} & -N^{ikm}\displaystyle\int_\Delta u^k_{00} \end{pmatrix} \begin{pmatrix} G^{mj}_{uu} & G^{mj}_{bu} \\ G^{mj}_{ub} & G^{mj}_{bb} \end{pmatrix} = \delta^{ij}\delta(\tau-\tau') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$f_{10}(\mathbf{k};\tau)$ solution

$$u^i_{10}(\mathbf{k};\tau) = \int_{-\infty}^{\tau} d\tau_1 \; G^{ij}_{uu}(\mathbf{k};\tau,\tau_1)\left[-D^{jk}(\mathbf{k})\frac{\overline{D}u^k_{00}(\mathbf{k};\tau_1)}{DT_{\mathrm{I}}} - D^{jk}(\mathbf{k})\left(\frac{\partial U^k}{\partial X^m} + \epsilon^{mkn}\Omega^n_0\right)u^m_{00}(\mathbf{k};\tau_1) + B^k\frac{\partial b^i_{00}(\mathbf{k};\tau_1)}{\partial X^k_{\mathrm{I}}} + D^{jk}(\mathbf{k})\frac{\partial B^k}{\partial X^m}b^m_{00}(\mathbf{k};\tau_1)\right]$$

$$+ \int_{-\infty}^{\tau} d\tau_1 \; G^{ij}_{ub}(\mathbf{k};\tau,\tau_1)\left[B^k\frac{\partial u^i_{00}(\mathbf{k};\tau_1)}{\partial X^k_{\mathrm{I}}} - D^{jk}(\mathbf{k})\frac{\partial B^k}{\partial X^m}u^m_{00}(\mathbf{k};\tau_1) - D^{jk}(\mathbf{k})\frac{\overline{D}b^k_{00}(\mathbf{k};\tau_1)}{DT_{\mathrm{I}}} + D^{jk}(\mathbf{k})\left(\frac{\partial U^k}{\partial X^m} + \epsilon^{mkn}\Omega_0\right)b^m_{00}(\mathbf{k};\tau_1)\right]$$

$$b^i_{10}(\mathbf{k};\tau) = \int_{-\infty}^{\tau} d\tau_1 \; G^{ij}_{bu}(\mathbf{k};\tau,\tau_1)\left[-D^{jk}(\mathbf{k})\frac{\overline{D}u^k_{00}(\mathbf{k};\tau_1)}{DT_{\mathrm{I}}} - D^{jk}(\mathbf{k})\left(\frac{\partial U^k}{\partial X^m} + \epsilon^{mkn}\Omega^n_0\right)u^m_{00}(\mathbf{k};\tau_1) + B^k\frac{\partial b^i_{00}(\mathbf{k};\tau_1)}{\partial X^k_{\mathrm{I}}} + D^{jk}(\mathbf{k})\frac{\partial B^k}{\partial X^m}b^m_{00}(\mathbf{k};\tau_1)\right]$$

$$+ \int_{-\infty}^{\tau} d\tau_1 \; G^{ij}_{bb}(\mathbf{k};\tau,\tau_1)\left[B^k\frac{\partial u^i_{00}(\mathbf{k};\tau_1)}{\partial X^k_{\mathrm{I}}} - D^{jk}(\mathbf{k})\frac{\partial B^k}{\partial X^m}u^m_{00}(\mathbf{k};\tau_1) - D^{jk}(\mathbf{k})\frac{\overline{D}b^k_{00}(\mathbf{k};\tau_1)}{DT_{\mathrm{I}}} + D^{jk}(\mathbf{k})\left(\frac{\partial U^k}{\partial X^m} + \epsilon^{mkn}\Omega_0\right)b^m_{00}(\mathbf{k};\tau_1)\right]$$

Turbulent electromotive force (EMF)

$$E_{\rm M}^i \equiv \epsilon^{ijk} \langle u'^j b'^k \rangle = \epsilon^{ijk} \int d\mathbf{k} \ \langle u^j(\mathbf{k};\tau) b^k(\mathbf{k}';\tau) \rangle / \delta(\mathbf{k}+\mathbf{k}').$$

$$\langle u^j b^k \rangle = \langle u_{00}^j b_{00}^k \rangle + \langle u_{01}^j b_{00}^k \rangle + \langle u_{00}^j b_{01}^k \rangle + \delta \langle u_{10}^j b_{00}^k \rangle + \delta \langle u_{00}^j b_{10}^k \rangle + \cdots .$$

α effect | Turb. Mag. Diffusivity | Turb. Pumping | Cross-helicity effect

$$\langle \mathbf{u}' \times \mathbf{b}' \rangle = \alpha \mathbf{B} - (\beta + \zeta) \nabla \times \mathbf{B} - (\nabla \zeta) \times \mathbf{B} + \gamma \nabla \times \mathbf{U}$$

$$I\{A, B\} = \int d\mathbf{k} \int_{-\infty}^{\tau} d\tau_1 A(k;\tau,\tau_1) B(k;\tau,\tau_1)$$

Cross interaction mediated by G_{ub} and G_{bu}

$$\alpha = \frac{1}{3} \left[-I\{G_{bb}, H_{uu}\} + I\{G_{uu}, H_{bb}\} - I\{G_{bu}, H_{ub}\} + I\{G_{ub}, H_{bu}\} \right]$$

$$\beta = \frac{1}{3} \left[I\{G_{bb}, Q_{uu}\} + I\{G_{uu}, Q_{bb}\} - I\{G_{bu}, Q_{ub}\} - I\{G_{ub}, Q_{bu}\} \right]$$

$$\zeta = \frac{1}{3} \left[I\{G_{bb}, Q_{uu}\} - I\{G_{uu}, Q_{bb}\} + I\{G_{bu}, Q_{ub}\} - I\{G_{ub}, Q_{bu}\} \right]$$

$$\gamma = \frac{1}{3} \left[I\{G_{bb}, Q_{ub}\} + I\{G_{uu}, Q_{bu}\} - I\{G_{bu}, Q_{uu}\} - I\{G_{ub}, Q_{bb}\} \right]$$

Dynamo due to the cross-interaction responses

Torsional cross correlations

$$\alpha_X = \frac{1}{3}\left[-I\{G_{bu}, H_{ub}\} + I\{G_{ub}, H_{bu}\}\right] \longrightarrow \alpha_X = -\tau_{bu}\langle \mathbf{u}' \cdot \mathbf{j}' \rangle + \tau_{ub}\langle \boldsymbol{\omega}' \cdot \mathbf{b}' \rangle$$

$$\beta_X = \frac{1}{3}\left[-I\{G_{bu}, Q_{ub}\} - I\{G_{ub}, Q_{bu}\}\right] \longrightarrow \beta_X = -(\tau_{bu} + \tau_{ub})\langle \mathbf{u}' \cdot \mathbf{b}' \rangle$$

$$\zeta_X = \frac{1}{3}\left[+I\{G_{bu}, Q_{ub}\} - I\{G_{ub}, Q_{bu}\}\right] \longrightarrow \zeta_X = (\tau_{bu} - \tau_{ub})\langle \mathbf{u}' \cdot \mathbf{b}' \rangle$$

$$\gamma_X = \frac{1}{3}\left[-I\{G_{bu}, Q_{uu}\} - I\{G_{ub}, Q_{bb}\}\right] \longrightarrow \gamma_X = -\tau_{bu}\langle \mathbf{u}'^2 \rangle - \tau_{ub}\langle \mathbf{b}'^2 \rangle$$

- **Vanish if** $G_{ub} = G_{bu} = 0$ No cross-interaction response effects

- **Finite cross-interaction responses with same timescales**

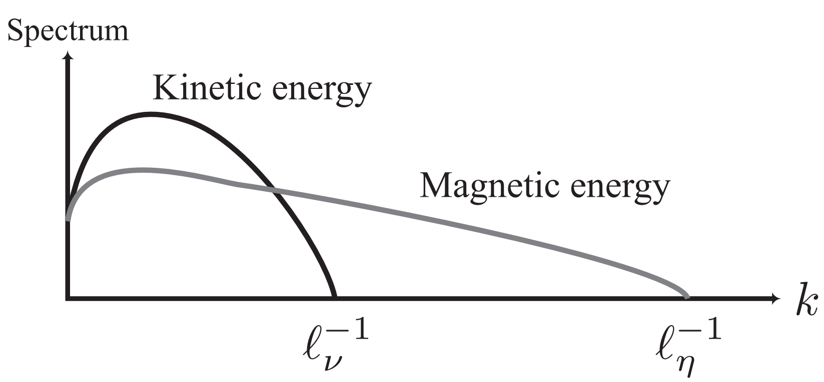
$G_{ub} = G_{bu} \neq 0$

$$\alpha_X = \tau_{bu}(-\langle \mathbf{u}' \cdot \mathbf{j}' \rangle + \langle \boldsymbol{\omega}' \cdot \mathbf{b}' \rangle) \qquad \tau_{bu} = \tau_{ub}$$
$$= \tau_{bu}\nabla \cdot \langle \mathbf{u}' \times \mathbf{b}' \rangle$$

EMF flux across the boundary

- **Timescale difference in responses** $G_{ub} \neq G_{bu}$

$\tau_{bu} \neq \tau_{ub}$ $\quad Pm \gg 1$ or $Pm \ll 1$



- **Non-equilibrium properties** $H_{ub}(\tau, \tau_1) \neq H_{ub}(\tau_1, \tau)$

Present talk

Cross-interaction α_X

$$\alpha_{\mathrm{X}} = -\frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1 \; G_{bu}(k,\mathbf{X};\tau,\tau_1,T) H_{ub}(k,\mathbf{X};\tau,\tau_1,T) + \frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1 \; G_{bu}(k,\mathbf{X};\tau,\tau_1,T) H_{bu}(k,\mathbf{X};\tau,\tau_1,T)$$

Helical functions satisfy $H_{bu}(\tau,\tau_1) = H_{ub}(\tau_1,\tau)$

$$\alpha_{\mathrm{X}} = -\frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1 \; G_{bu}(k,\mathbf{X};\tau,\tau_1,T) H_{ub}(k,\mathbf{X};\tau,\tau_1,T) + \frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1 \; G_{bu}(k,\mathbf{X};\tau,\tau_1,T) H_{ub}(k,\mathbf{X};\tau_1,\tau,T)$$

Symmetric and anti-symmetric parts of H_{ub}
with respect to the exchange of time variables

$$H_{ub}^{(\mathrm{S})}(\tau,\tau_1) = \frac{1}{2}\left(H_{ub}(\tau,\tau_1) + H_{ub}(\tau_1,\tau)\right)$$

$$H_{ub}^{(\mathrm{A})}(\tau,\tau_1) = \frac{1}{2}\left(H_{ub}(\tau,\tau_1) - H_{ub}(\tau_1,\tau)\right)$$ Non-equilibrium effect

$$H_{ub}(\tau,\tau_1) \neq H_{ub}(\tau_1,\tau)$$

$$\alpha_{\mathrm{X}} = -\frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1 \ [G_{ub}(k,\mathbf{X};\tau,\tau_1,T) + G_{bu}(k,\mathbf{X};\tau,\tau_1,T)] H_{ub}^{(\mathrm{A})}(k,\mathbf{X};\tau,\tau_1,T)$$

$$+\frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1 \ [G_{ub}(k,\mathbf{X};\tau,\tau_1,T) - G_{bu}(k,\mathbf{X};\tau,\tau_1,T)] H_{ub}^{(\mathrm{S})}(k,\mathbf{X};\tau_1,\tau,T)$$

Non-equilibrium alpha

$$\alpha_{\mathrm{neq}} = -\frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1 \ [G_{ub}(k,\mathbf{X};\tau,\tau_1,T) + G_{bu}(k,\mathbf{X};\tau,\tau_1,T)] H_{ub}^{(\mathrm{A})}(k,\mathbf{X};\tau,\tau_1,T)$$

Simple model

$$\mathcal{G}(\tau,\tau_1) = G_{ub}(\tau,\tau_1) + G_{bu}(\tau,\tau_1)$$ Independent of *k*

$$\alpha_{\mathrm{neq}} = -\frac{1}{3}\int_{-\infty}^{\tau} d\tau_1 \ \mathcal{G}(\tau,\tau_1) \langle \mathbf{u}'_{00} \cdot \mathbf{j}'_{00} \rangle^{(\mathrm{A})}(\mathbf{x};\tau,\tau_1)$$

where $\langle \mathbf{u}'_{00} \cdot \mathbf{j}'_{00} \rangle^{(\mathrm{A})}(\mathbf{x};\tau,\tau_1) = \frac{1}{2}\left[\langle \mathbf{u}'_{00}(\mathbf{x};\tau) \cdot \mathbf{j}'_{00}(\mathbf{x};\tau_1)\rangle - \langle \mathbf{u}'_{00}(\mathbf{x};\tau_1) \cdot \mathbf{j}'_{00}(\mathbf{x};\tau)\rangle\right]$

Memory effect is crucial since $\langle \mathbf{u}'_{00} \cdot \mathbf{j}'_{00} \rangle^{(\mathrm{A})}(\mathbf{x};\tau,\tau) = 0$

$$\begin{pmatrix} \dfrac{\partial}{\partial \tau} + \nu k^2 & 0 \\ 0 & \dfrac{\partial}{\partial \tau} + \eta k^2 \end{pmatrix} \begin{pmatrix} G_{uu}^{ij} & G_{bu}^{ij} \\ G_{ub}^{ij} & G_{bb}^{ij} \end{pmatrix}$$

$$+i \begin{pmatrix} -2M^{ikm} \int_\Delta u_{00}^k & 2M^{ikm} \int_\Delta b_{00}^k \\ N^{ikm} \int_\Delta b_{00}^k & -N^{ikm} \int_\Delta u_{00}^k \end{pmatrix} \begin{pmatrix} G_{uu}^{mj} & G_{bu}^{mj} \\ G_{ub}^{mj} & G_{bb}^{mj} \end{pmatrix} = \delta^{ij} \delta(\tau - \tau') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The response functions $\boldsymbol{G}_{ub}$ must be an odd function of $\mathbf{b}_{00}$

$\langle \mathbf{u}'_{00} \cdot \mathbf{j}'_{00} \rangle$ pure scalar → G_{ub} skew

Dynamic quantity that is odd in $\mathbf{b}_{00}$ and skew → Cross helicity $\langle \mathbf{u}'_{00} \cdot \mathbf{b}'_{00} \rangle$

$$\alpha_{\rm neq} \sim -\frac{2}{3} \int_{-\infty}^{\tau} d\tau_1 \Upsilon^{(\rm S)}(\mathbf{x}; \tau, \tau_1) \langle \mathbf{u}'_{00} \cdot \mathbf{j}'_{00} \rangle^{(\rm A)}(\mathbf{x}; \tau, \tau_1)$$

where $$\Upsilon(\mathbf{x}; \tau, \tau_1) \equiv \frac{\langle \mathbf{u}'_{00}(\mathbf{x}; \tau) \cdot \mathbf{b}'_{00}(\mathbf{x}; \tau_1) \rangle}{\sqrt{\langle {u'_{00}}^2 \rangle (\mathbf{x}; \tau)} \sqrt{\langle {b'_{00}}^2 \rangle (\mathbf{x}; \tau_1)}}$$ Normalised cross helicity

$$\Upsilon^{(\rm S)}(\mathbf{x}; \tau, \tau_1) = \frac{1}{2} [\Upsilon(\mathbf{x}; \tau, \tau_1) + \Upsilon(\mathbf{x}; \tau_1, \tau)]$$

→ $$\alpha_{\rm neq} \sim -\frac{2}{3} \int_{-\infty}^{\tau} d\tau_1 \left(\Upsilon^{(\rm S)}(\mathbf{x}; \tau, \tau_1) \right)^2 \langle \mathbf{u}'_{00} \cdot \boldsymbol{\omega}'_{00} \rangle^{(\rm A)}(\mathbf{x}; \tau, \tau_1)$$

Non-equilibrium alpha

$$\alpha_{\rm neq} \sim -\frac{2}{3}\int_{-\infty}^{\tau} d\tau_1 \left(\Upsilon^{(\rm S)}(\mathbf{x};\tau,\tau_1)\right)^2 \langle \mathbf{u}'_{00}\cdot\boldsymbol{\omega}'_{00}\rangle^{(\rm A)}(\mathbf{x};\tau,\tau_1)$$

Cross helicity Non-equilibrium helicity

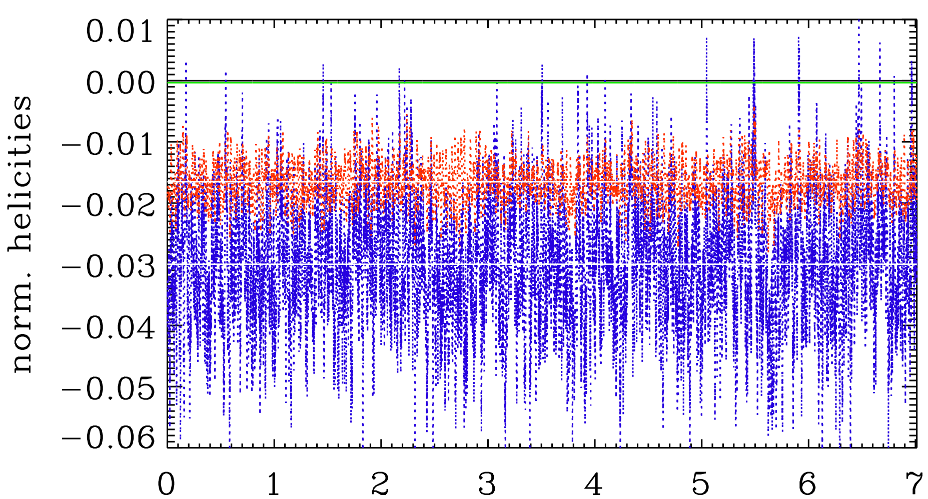
Validation of $\alpha_{\rm neq}$ by DNSs

$$\mathbf{g} \parallel \nabla\rho \parallel \mathbf{B}_0 \parallel \omega_{\rm F}$$

$\boldsymbol{\Omega}_{\rm F} \parallel \nabla\rho$ ⟶ Kinetic helicity

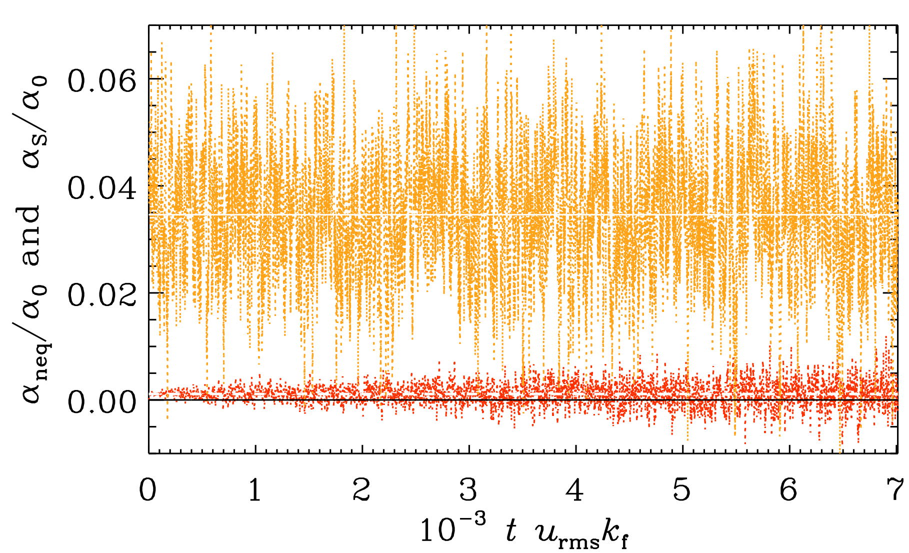
$\mathbf{B} \parallel \nabla\rho$ ⟶ Cross helicity

Co-existence of kinetic and cross helicities



	$\frac{g}{c_s^2 k_1}$	$\frac{v_{A0}}{c_s}$	$\frac{\langle \mathbf{u}'\cdot\mathbf{b}'\rangle}{\sqrt{\langle u'^2\rangle\langle b'^2\rangle}}$	$\frac{\langle \mathbf{u}'\cdot\mathbf{w}'\rangle}{\sqrt{\langle u'^2\rangle\langle w'^2\rangle}}$	$\frac{\langle \mathbf{b}'\cdot\mathbf{j}'\rangle}{\sqrt{\langle u'^2\rangle\langle w'^2\rangle}}$	$\frac{\alpha_{\rm neq}}{\alpha_0}$	$\frac{\alpha_{\rm S}}{\alpha_0}$	$\frac{u_{\rm rms}}{c_s}$	$\frac{v_A^{\rm rms}}{c_s}$
C	0.5	0.01	-9.8×10^{-3}	-1.6×10^{-2}	-2.0×10^{-4}	7.8×10^{-4}	1.8×10^{-2}	0.10	0.03
A	1.0	0.01	-1.7×10^{-2}	-3.0×10^{-2}	-3.3×10^{-4}	1.1×10^{-3}	3.5×10^{-2}	0.11	0.04
D	2.0	0.01	-2.0×10^{-2}	-3.6×10^{-2}	-2.8×10^{-4}	6.1×10^{-4}	4.1×10^{-2}	0.16	0.04
E	0.5	0.10	-5.5×10^{-2}	-1.9×10^{-2}	-6.2×10^{-4}	-5.6×10^{-3}	1.5×10^{-2}	0.08	0.07
B	1.0	0.10	-5.3×10^{-2}	-3.2×10^{-2}	-1.2×10^{-2}	2.3×10^{-3}	1.8×10^{-2}	0.09	0.12

TABLE 1. Summary of the simulation results for Runs A–E.



	$k_{\rm f}$	$\frac{\langle \mathbf{u}'\cdot\mathbf{b}'\rangle}{\sqrt{\langle u'^2\rangle\langle b'^2\rangle}}$	$\frac{\langle \mathbf{u}'\cdot\mathbf{w}'\rangle}{\sqrt{\langle u'^2\rangle\langle w'^2\rangle}}$	$\frac{\langle \mathbf{b}'\cdot\mathbf{j}'\rangle}{\sqrt{\langle u'^2\rangle\langle w'^2\rangle}}$	$\frac{\alpha_{\rm neq}}{\alpha_0}$	$\frac{\alpha_{\rm S}}{\alpha_0}$	$\frac{u_{\rm rms}}{c_s}$	$\frac{v_A^{\rm rms}}{c_s}$
A	30	-1.7×10^{-2}	-3.0×10^{-2}	-3.3×10^{-4}	1.1×10^{-3}	3.5×10^{-2}	0.11	0.04
A2	10	-1.3×10^{-1}	-1.2×10^{-1}	1.3×10^{-3}	-1.7×10^{-2}	6.9×10^{-2}	0.12	0.12
A3	3	-6.4×10^{-2}	-2.1×10^{-1}	-3.0×10^{-2}	-6.0×10^{-3}	5.5×10^{-2}	0.19	0.09

TABLE 2. Summary of the simulation results for Runs A, A2, and A3.

Summary of non-equilibrium effects on dynamos

Cross-interaction response functions

$$G_{ub}, \quad G_{bu}$$

Torsional cross correlations

$$\langle \mathbf{u}' \cdot \mathbf{j}' \rangle, \quad \langle \boldsymbol{\omega}' \cdot \mathbf{b}' \rangle$$

Non-equilibrium properties of turbulence

$$\langle \mathbf{u}'_{00}(\mathbf{x};\tau) \cdot \mathbf{j}'_{00}(\mathbf{x};\tau_1) \rangle \neq \langle \mathbf{u}'_{00}(\mathbf{x};\tau_1) \cdot \mathbf{j}'_{00}(\mathbf{x};\tau) \rangle$$

Co-existence of kinetic and cross helicities

$$\langle \mathbf{u}' \cdot \mathbf{b}' \rangle, \quad \langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle$$

Beyond heuristic modelling

Mean-field equations in compressible MHD

Yokoi, N., J. Plasma Phys. **84**, 735840501 & 775840603 (2018a,b)

Density

$$\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho} \mathbf{U}) = -\nabla \cdot \langle \rho' \mathbf{u}' \rangle$$

Turb. mass flux

Momentum

$$\frac{\partial}{\partial t} \overline{\rho} U^\alpha + \frac{\partial}{\partial x^a} \overline{\rho} U^a U^\alpha$$

$$= -(\gamma_0 - 1) \frac{\partial}{\partial x^\alpha} \overline{\rho} Q + \frac{\partial}{\partial x^\alpha} \mu S^{a\alpha} + (\mathbf{J} \times \mathbf{B})^\alpha$$

$$- \frac{\partial}{\partial x^\alpha} \left(\overline{\rho} \langle u'^a u'^\alpha \rangle - \frac{1}{\mu_0} \langle b'^a b'^\alpha \rangle + U^a \langle \rho' u'^\alpha \rangle + U^\alpha \langle \rho' u'^a \rangle \right) + R_U^\alpha$$

Reynolds stress

Turb. Maxwell stress

Internal energy

Turb. energy flux

Turb. mass-energy correl.

$$\frac{\partial}{\partial t} \overline{\rho} Q + \nabla \cdot (\overline{\rho} \mathbf{U} Q) = \nabla \cdot \left(\frac{\kappa}{C_V} \nabla Q \right) - \nabla \cdot (\overline{\rho} \langle q' \mathbf{u}' \rangle + Q \langle \rho' \mathbf{u}' \rangle + \mathbf{U} \langle \rho' q' \rangle)$$

$$- (\gamma_0 - 1) \left(\overline{\rho} Q \nabla \cdot \mathbf{U} + \overline{\rho} \langle q' \nabla \cdot \mathbf{u}' \rangle + Q \langle \rho' \nabla \cdot \mathbf{u}' \rangle \right) + R_Q$$

Turb. energy dilatation

Turb. mass dilatation

Magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B} + \langle \mathbf{u}' \times \mathbf{b}' \rangle) + \eta \nabla^2 \mathbf{B}$$

Turb. electromotive force

Some main results of theoretical analysis

Reynolds and turbulent Maxwell stress

eddy viscosity — inhomogeneous helicity

$$\langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}' \rangle_{\mathrm{D}} = -\nu_{\mathrm{K}} \boldsymbol{\mathcal{S}} + \nu_{\mathrm{M}} \boldsymbol{\mathcal{M}} + \eta_H \nabla H \boldsymbol{\Omega}_* + \cdots$$

D: deviatoric part — cross helicity

mean velocity strain $\boldsymbol{\mathcal{S}} = \nabla \mathbf{U} + (\nabla \mathbf{U})^\dagger$ mean magnetic-field strain $\boldsymbol{\mathcal{M}} = \nabla \mathbf{B} + (\nabla \mathbf{B})^\dagger$

$\boldsymbol{\Omega}_*$: absolute mean vorticity (mean vorticity + rotation)

Turbulent electromotive force

Turb. Mag. Diffusivity — Alpha effect

$$\langle \mathbf{u}' \times \mathbf{b}' \rangle = -(\beta + \zeta) \nabla \times \mathbf{B} + \gamma \nabla \times \mathbf{U} + \alpha \mathbf{B} + (\nabla \zeta) \times \mathbf{B}$$

Cross-helicity effect — Magnetic pumping

$$- \chi_\rho \nabla \overline{\rho} \times \mathbf{B} - \chi_Q \nabla Q \times \mathbf{B} - \chi_D \frac{D\mathbf{U}}{Dt} \times \mathbf{B}$$

Compressibility

Turbulent mass flux

$$\langle \rho' \mathbf{u}' \rangle = -\kappa_{\overline{\rho}} \nabla \overline{\rho} - \kappa_Q \nabla Q - \kappa_D \frac{D\mathbf{U}}{DT} - \kappa_B \mathbf{B}$$

Turbulent internal-energy flux

$$\langle q' \mathbf{u}' \rangle = -\eta_Q \nabla Q - \eta_{\overline{\rho}} \nabla \overline{\rho} - \eta_B \mathbf{B}$$

+ Non-equilibrium effects

Transport coefficients

Mean fields

$\overline{\rho}$

$\mathbf{U}$

$\mathbf{B}$

Q

One-point
turbulent statistical quantities

represented by

$$\left[K_\rho \equiv \langle \rho'^2 \rangle\right]$$

$$K \equiv \frac{1}{2} \langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle$$

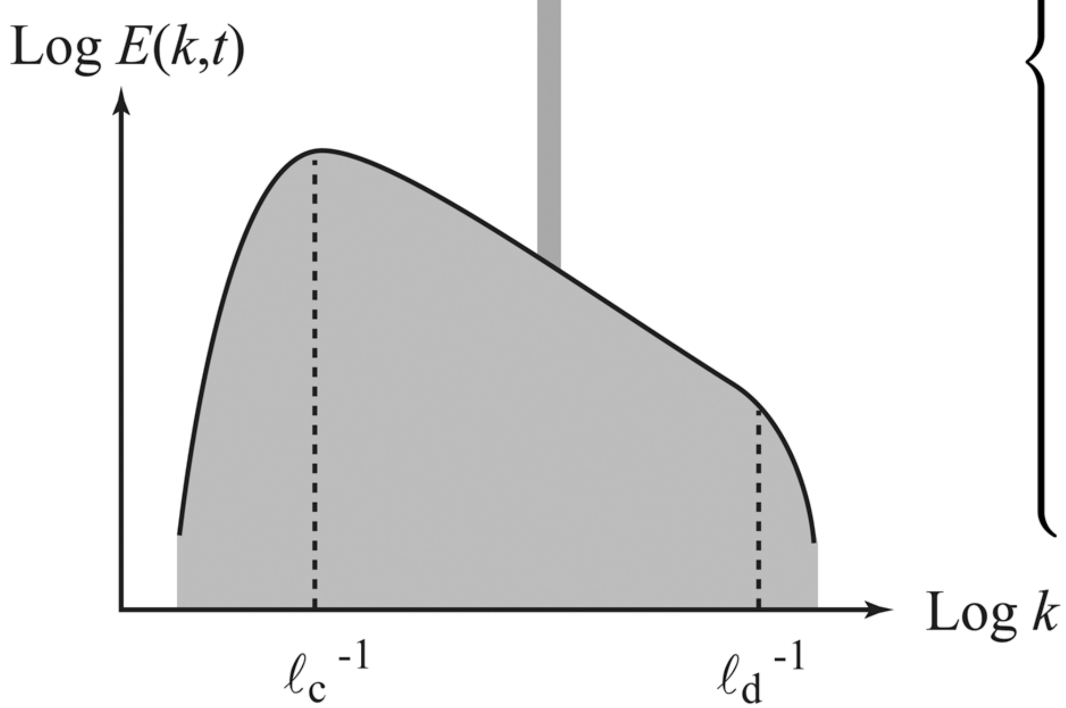
$$\varepsilon \equiv \nu \left\langle \left(\frac{\partial u'_a}{\partial x_b} \right)^2 \right\rangle + \eta \left\langle \left(\frac{\partial b'_a}{\partial x_b} \right)^2 \right\rangle$$

$$W \equiv \langle \mathbf{u}' \cdot \mathbf{b}' \rangle$$

$$\left[K_{\mathrm{R}} \equiv \frac{1}{2} \langle \mathbf{u}'^2 - \mathbf{b}'^2 \rangle\right]$$

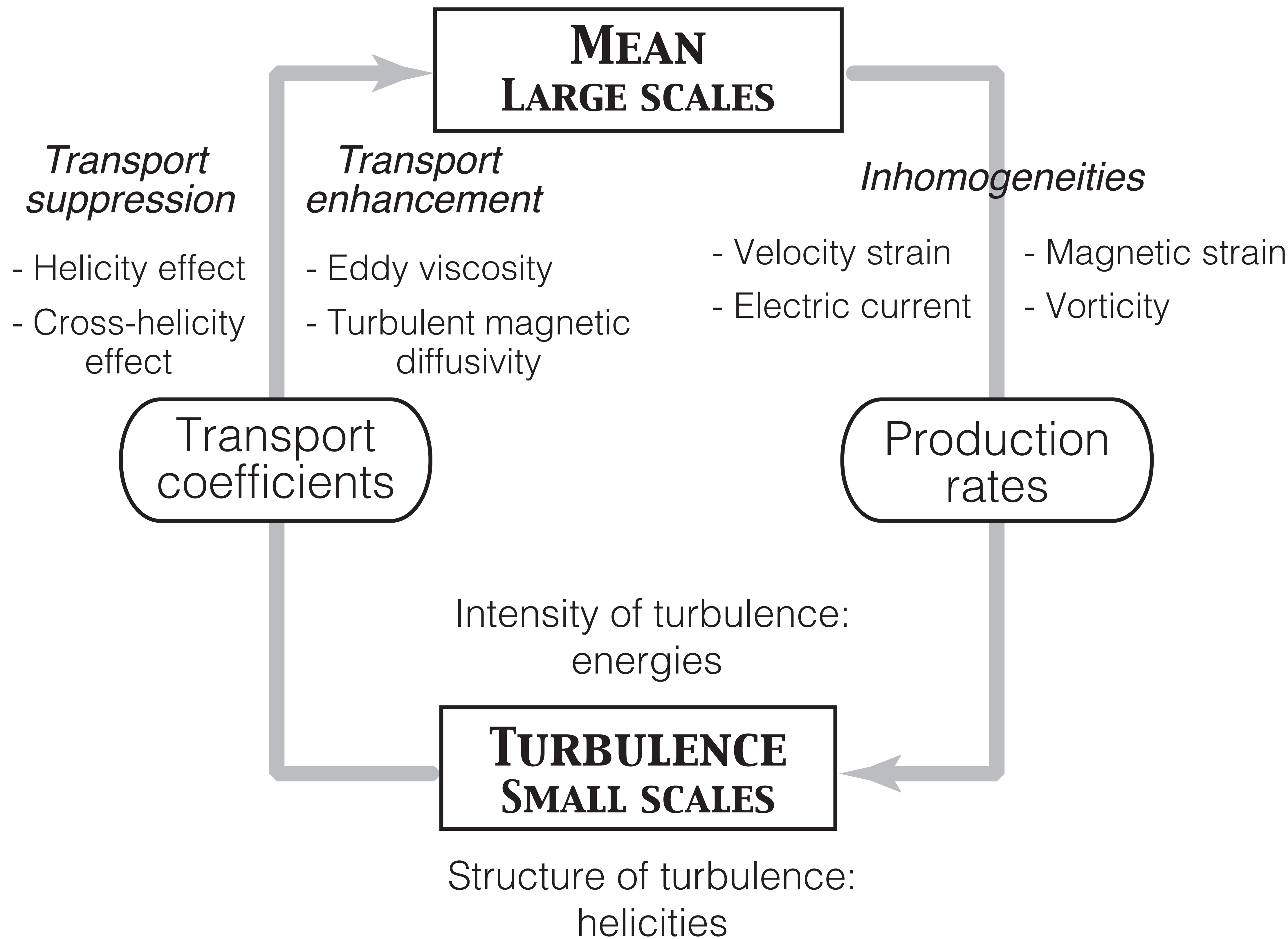
$$\left[\varepsilon_W \equiv (\nu + \eta) \left\langle \frac{\partial u'_a}{\partial x_b} \frac{\partial b'_a}{\partial x_b} \right\rangle\right]$$

$$\left[H \equiv \langle -\mathbf{u}' \cdot \boldsymbol{\omega}' + \mathbf{b}' \cdot \mathbf{j}' \rangle\right]$$

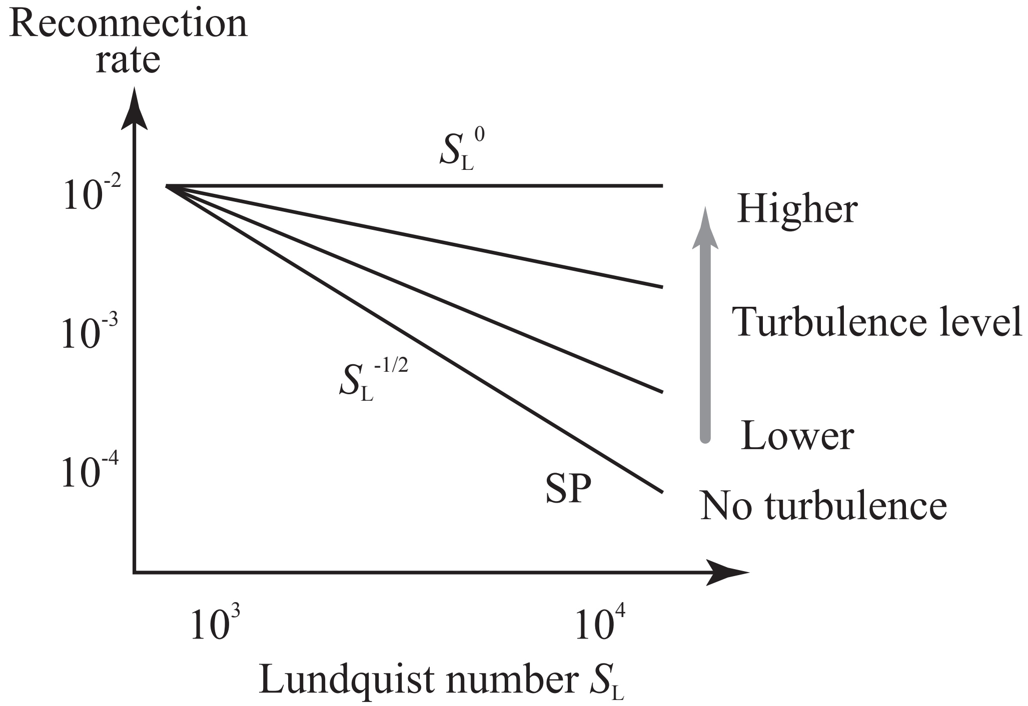


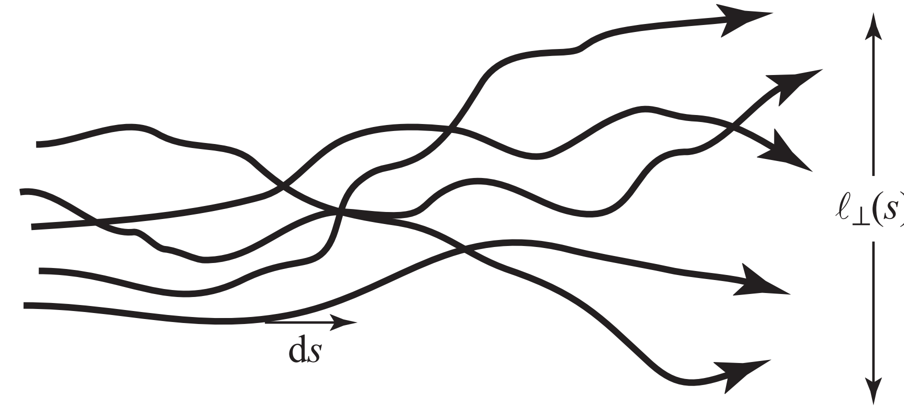
Beyond the heuristic modelling

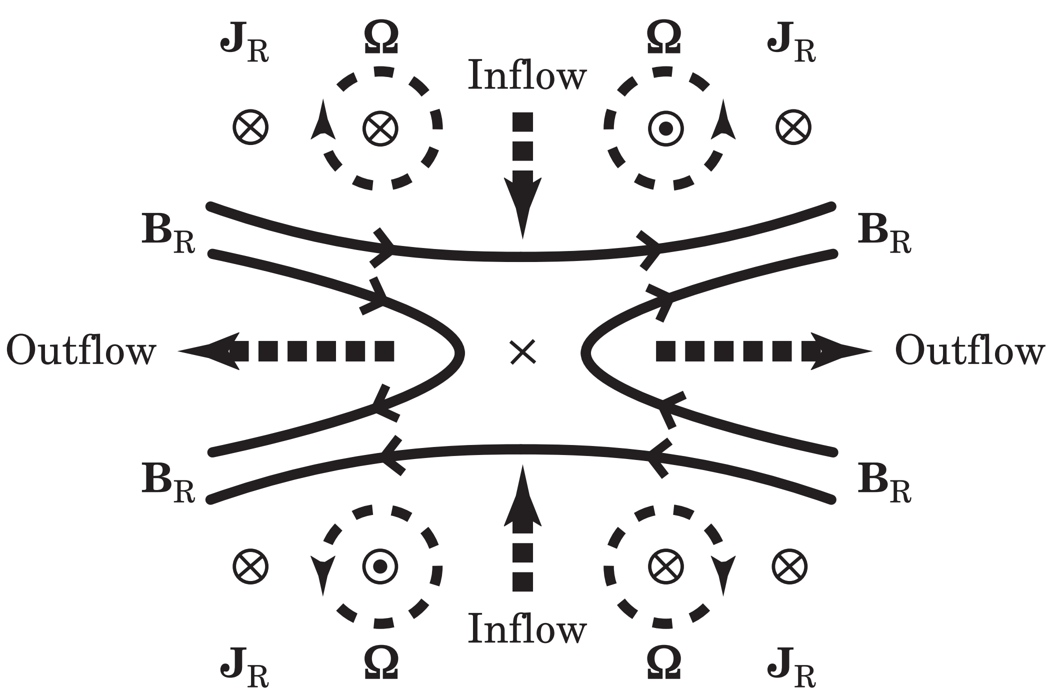
- Model structures
- Transport coefficients

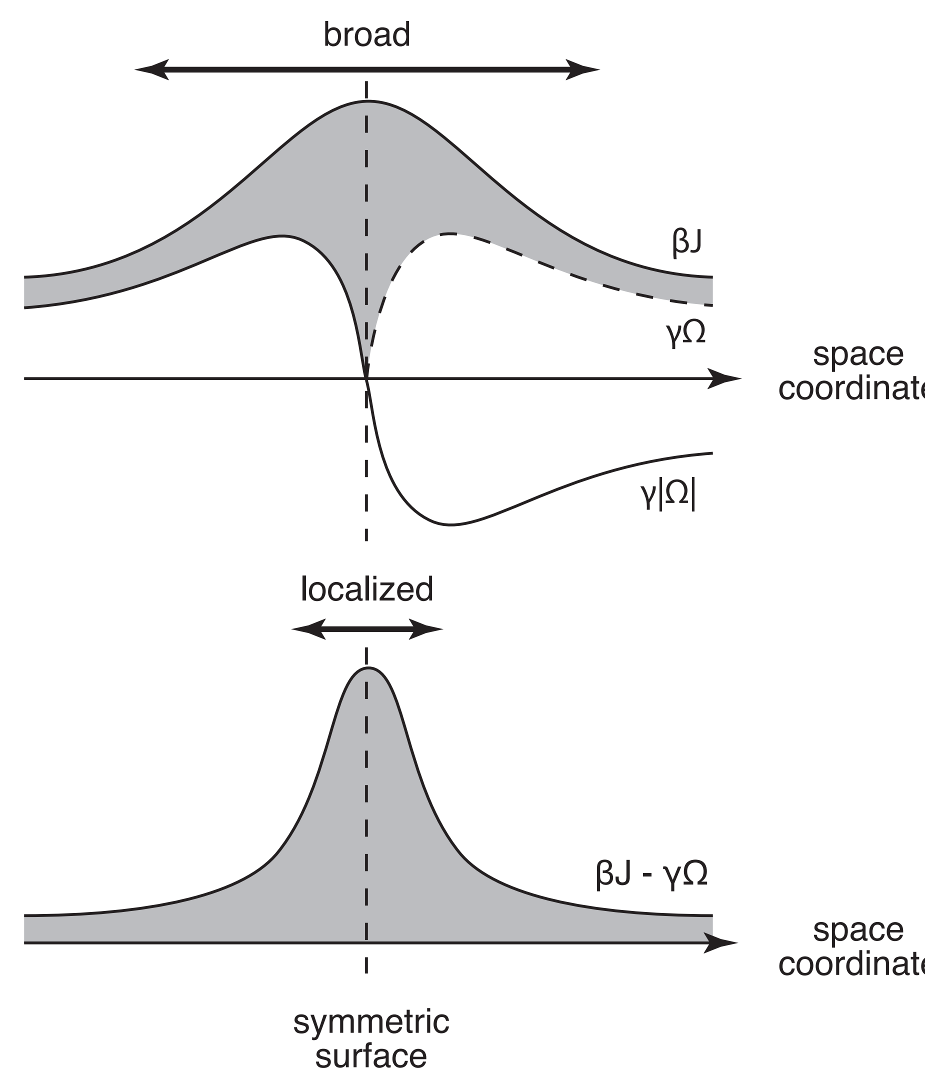

MEAN
LARGE SCALES
Transport suppression
- Helicity effect
- Cross-helicity effect
Transport enhancement
- Eddy viscosity
- Turbulent magnetic diffusivity
Inhomogeneities
- Velocity strain
- Electric current
- Magnetic strain
- Vorticity
Transport coefficients
Production rates
Intensity of turbulence: energies
TURBULENCE
SMALL SCALES
Structure of turbulence: helicities

Turbulent magnetic reconnection


Reconnection rate
10^{-2}
10^{-3}
10^{-4}
S_L^{0}
$S_L^{-1/2}$
SP
Higher
Turbulence level
Lower
No turbulence
10^{3}
10^{4}
Lundquist number S_L


ds
$\ell_\perp(s)$


$\mathbf{J}_R$
$\mathbf{\Omega}$
Inflow
$\mathbf{\Omega}$
$\mathbf{J}_R$
$\mathbf{B}_R$
$\mathbf{B}_R$
Outflow
Outflow
$\mathbf{B}_R$
$\mathbf{B}_R$
Inflow
$\mathbf{J}_R$
$\mathbf{\Omega}$
$\mathbf{\Omega}$
$\mathbf{J}_R$

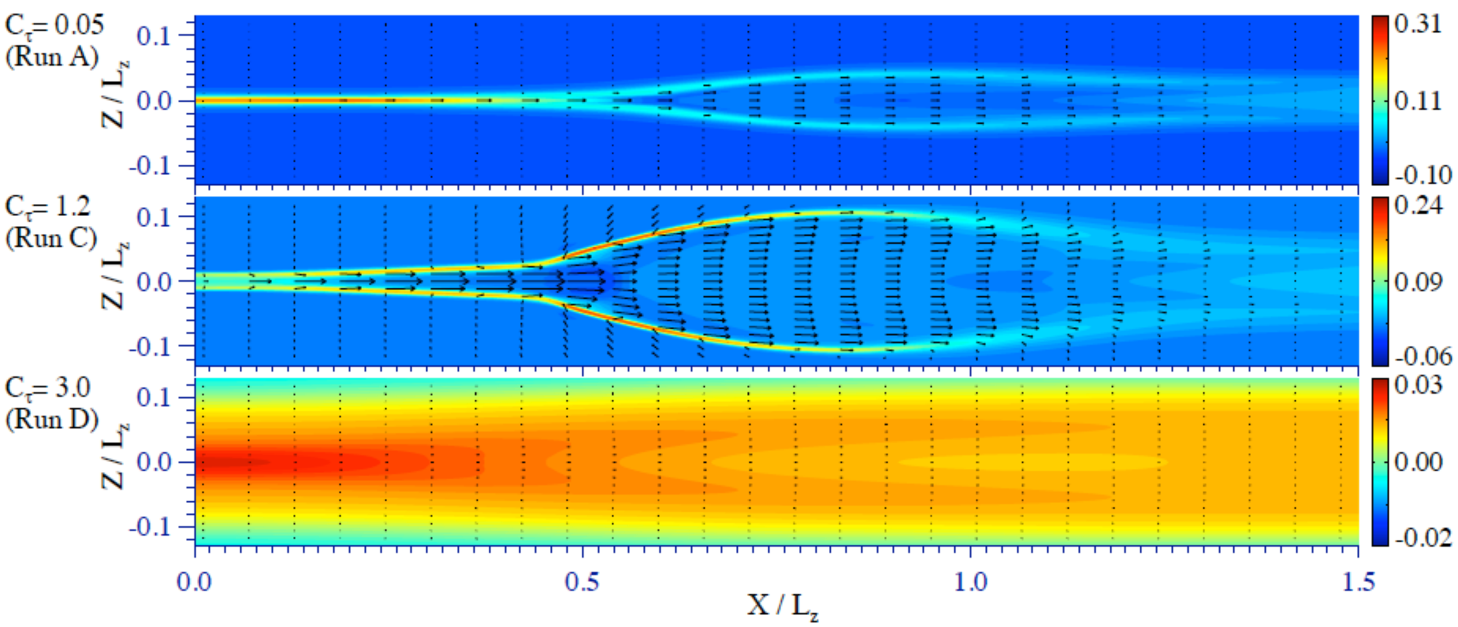

broad
βJ
γΩ
space coordinate
γ|Ω|
localized
βJ - γΩ
space coordinate
symmetric surface

Turbulence modeling approach to magnetic reconnection

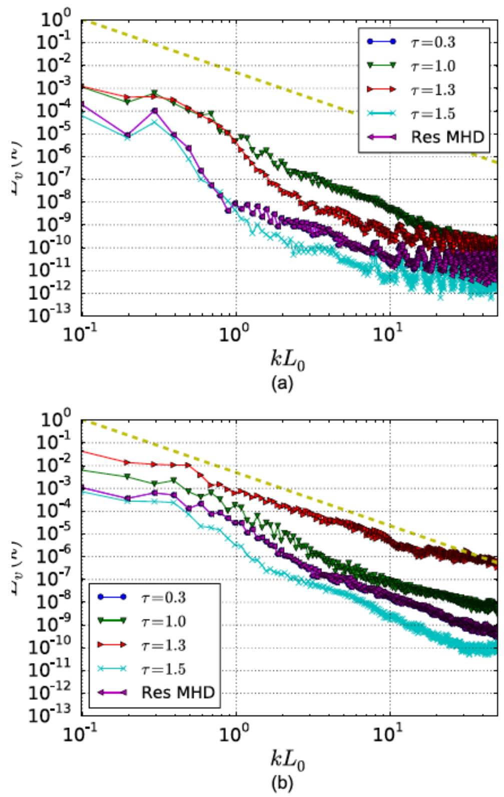
Yokoi & Hoshino, Phys. Plasmas **18**, 111208 (2011)

Spectral evolution

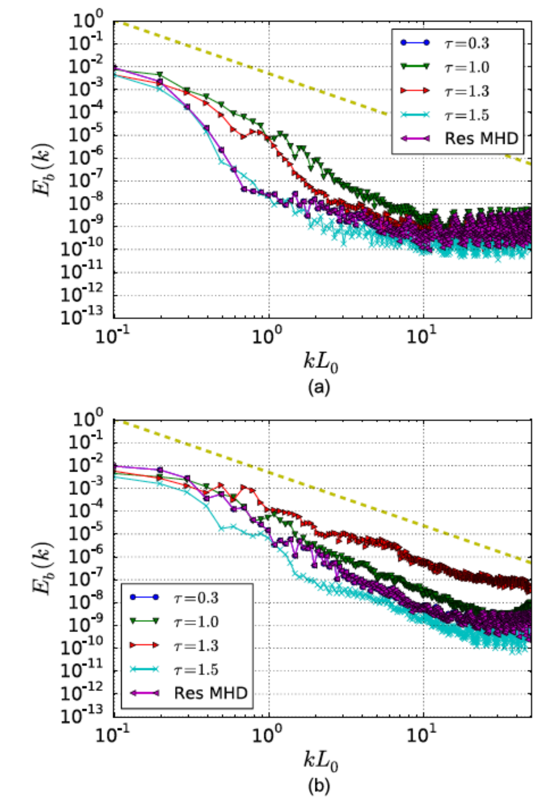
Electric-current and flow structures



Kinetic energy

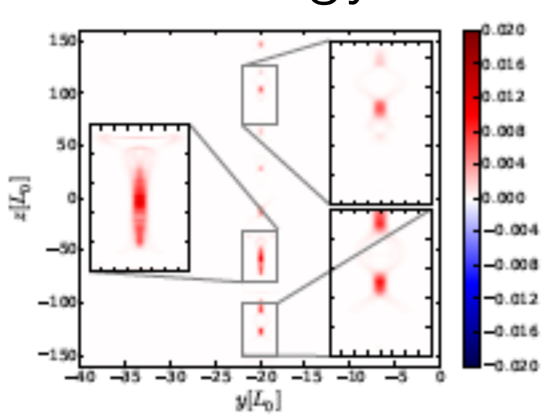


Magnetic energy

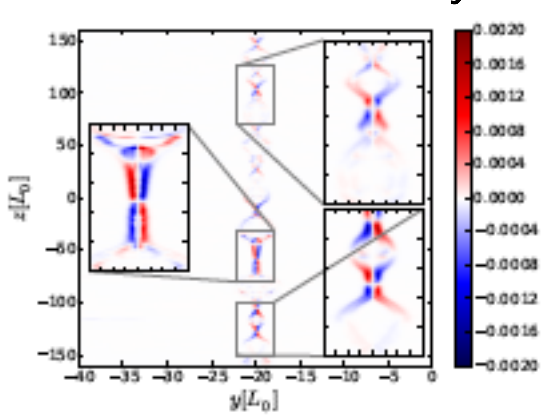


Higashimori, Yokoi & Hoshino, PRL **110**, 255001 (2013)

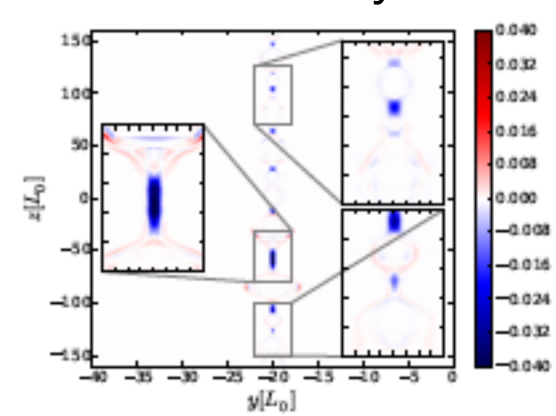
Energy



Cross helicity



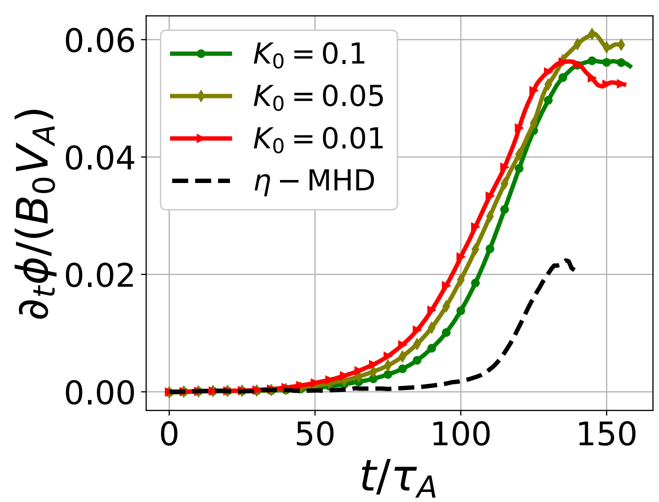
Helicity ← Guide field



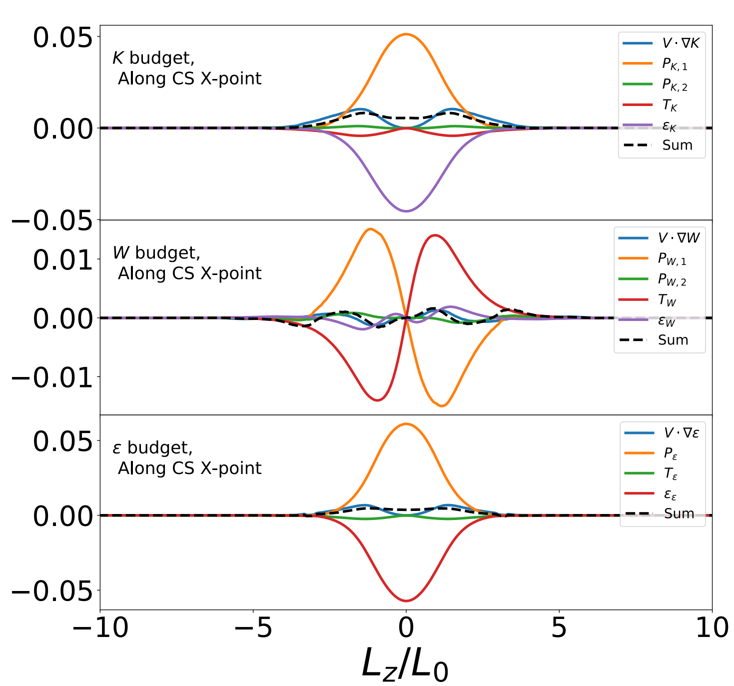
Widmer, Büchner & Yokoi, Phys. Plasmas **23**, 092304 (2016)

Self-consistent determination of turbulence times

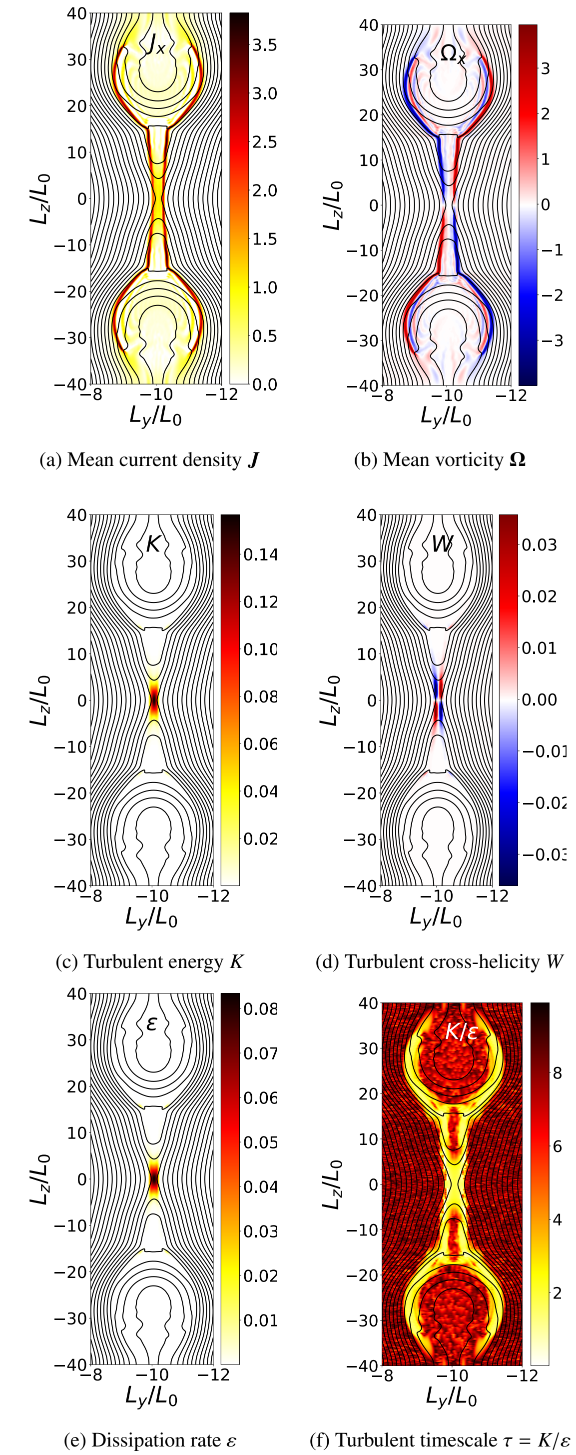
Widmer, Büchner, and Yokoi (2019) Phys. Plasmas, accepted on 25 Sep. 2019



Reconnection rate with
changing initial turbulent energy



Budget of turbulent energy *K*, cross helicity *W*,
and energy dissipation rate *ε*



Spatial distributions of mean
and turbulent quantities

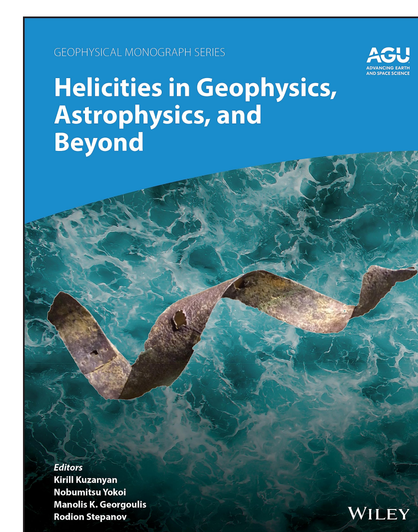
Advertisement:
Turbulence, Reconnection, Dynamos, Helicities…

Yokoi, N. “Turbulence, transport and reconnection,” Chap. 6 in **Topics in *Magnetohydrodynamic Topology, Reconnection and Stability Theory:* CISM International Centre for Mechanical Sciences 591 pp.177-265 (Springer, 2020)**
https://doi.org/10.1007/978-3-030-16343-3_6

Pouquet, A. & Yokoi, N. “Helical fluid and (Hall)-MHD turbulence: a brief review,” **Phil. Trans. Roy. Soc. A 380, 20210081 (2022)**
https://doi.org/10.1098/rsta.2021.0087

Yokoi, N. “Unappreciated cross-helicity effects in plasma physics: anti-diffusion effects in dynamo and momentum transport,” **Reviews of Modern Plasma Physics 7, 33 (2023)**
https://doi.org/10.1007/s41614-023-00133-4

Kuzanyan, K., Yokoi, N., Georgoulis, M. & Stepanov, R. (Eds.) *AGU Books 283:* ***Helicities in Geophysics, Astrophysics, and Beyond*** **(Wiley, 2024)**
https://doi.org/10.1002/9781119841715



Nobu Yokoi <nobyokoi@iis.u-tokyo.ac.jp>

Theory

Yokoi, N. "Turbulence, transport and reconnection," Chap. 6 in Topics in *Magnetohydrodynamic Topology, Reconnection and Stability Theory:* CISM International Centre for Mechanical Sciences 591 pp.177-265 (Springer, 2020)
https://doi.org/10.1007/978-3-030-16343-3_6

Helicity

Pouquet, A. & Yokoi, N. "Helical fluid and (Hall)-MHD turbulence: a brief review," Phil. Trans. Roy. Soc. A 380, 20210081-1-18 (2022)
https://doi.org/10.1098/rsta.2021.0087

Yokoi, N. "Transports in helical fluid turbulence," pp.25-50, in Kuzanyan, Yokoi, Georgoulis & Stepanov (eds.) *AGU Book: Helicities in Geophysics, Astrophysics and Beyond* (Wiley, 2023)
https://doi.org/10.1002/9781119841715

Yokoi, N. & Brandenburg, A. "Global flow generation by inhomogeneous helicity," (2016) Phys. Rev. E. 93, 033125-1-14 (2016)
https://doi.org/10.1103/PhysRevE.93.033125

Dynamos

Yokoi, N. "Cross-helicity and related dynamo," Geophys. Astrophys. Fluid Dyn. 107, 114-184 (2013)
https://doi.org/10.1080/03091929.2012.754022

Yokoi, N. "Unappreciated cross-helicity effects in plasma physics: Anti-diffusion effects in dynamo and momentum transport," Rev. Mod. Plasma Phys. 7, 33-1-98 (2023)
https://doi.org/10.1007/s41614-023-00133-4

Convection

Yokoi, N., Masada, Y. & Takiwaki, T. “Modelling stellar convective transport with plumes: I. Non-equilibrium turbulence effect in double-averaging formulation,” MNRAS 516, 2718–2735 (2022)
https://doi.org/10.1093/mnras/stac1181

Yokoi, N. “Non-Equilibrium Turbulent Transport in Convective Plumes Obtained from Closure Theory,” Atmosphere 14, 1013-1-22 (2023)
https://doi.org/10.3390/atmos14061013

Strong compressibility

Electromotive force

Yokoi, N. “Electromotive force in strongly compressible magneto-hydrodynamic turbulence,” J. Plasma Phys. 84, 735840501-1-26 (2018)
https://doi.org/10.1017/S0022377818000727

Mass/Heat

Yokoi, N. “Mass and internal-energy transports in strongly compressible magnetohydrodynamic turbulence,” J. Plasma Phys. 84, 775840603-1-30 (2018)
https://doi.org/10.1017/S0022377818001228

Instability

Yokoi, N. & Tobias, S. M. “Magnetoclinicity Instability,” Prog. in Turb. IX, Springer Proceedings in Physics 267, 273-279 (2021)
https://doi.org/10.1080/03091929.2012.754022 (arXiv:2205.14453)